ALGEBRAIC CURVES AND THEIR MODULI
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Rough structure of my lectures:

- From parameters to moduli.
- Moduli functors and moduli spaces.
- Local study: deformation theory.
- Brill-Noether theory and moduli.

Historical highlights

- Riemann (1851)
- Brill and Noether (1873)
- Severi (1915)

Parameters

All schemes will be algebraic over $\mathbb{C}$.

NOTATION: $X_0, X_1, \dots$ denote homogeneous coordinates in a projective space,
$X, Y, Z, \dots$ denote coordinates in an affine space.

Algebraic varieties depend on parameters.
This is clear if we define them by means of equations in some (affine or projective) space, because one can vary the coefficients of the equations.
e.g. by moving the coefficients of their equation we parametrize nonsingular plane curves of degree d in $\mathbb{P}^2$ by the points of (an open subset of) a $\mathbb{P}^N$, where $N = \frac{d(d+3)}{2}$.

Less obviously, consider a nonsingular rational cubic curve $\mathcal{C} \subset \mathbb{P}^3$. Up to choice of coordinates it can be defined by the three quadric equations:

$$X_1X_3 - X_2^2 = X_0X_3 - X_1X_2 = X_0X_2 - X_1^2 = 0 \tag{1}$$

that I will write as

$$\mathrm{rk}\begin{pmatrix} X_0 & X_1 & X_2 \\ X_1 & X_2 & X_3 \end{pmatrix} \leq 1 \tag{2}$$

If we deform arbitrarily the coefficients of the equations (1) their intersection will consist of 8 distinct points. Bad choice!
Good choice: deform the entries of the matrix (2) to general linear forms in $X_0, \ldots, X_3$. This will correspond to ask that the corresponding family of subvarieties is flat, and will guarantee that we obtain twisted cubics again.

From parameters to moduli

Parameters are a naive notion. Moduli is a more refined notion: they are parameters of *isomorphism classes* of objects.

Typical example: the difference between parametrizing plane conics and parametrizing plane cubics.
Conics depend on 5 parameters but have no moduli, cubics depend on 9 parameters and have 1 modulus.

What does it mean that cubics have one modulus?
This has been an important discovery in the XIX century. It consists of the following steps:

- given 4 pairwise distinct points $P_i = (a_i, b_i) \in \mathbb{P}^1$, $i = 1, \dots, 4$ consider their cross ratio

  $$\lambda = \frac{(a_1 b_3 - a_3 b_1)(a_2 b_4 - a_4 b_2)}{(a_1 b_4 - a_4 b_1)(a_2 b_3 - a_3 b_2)}$$

  It is invariant under linear coordinate changes (direct computation) and takes all values $\neq 0, 1$.
  Moreover for a general 4-tuple λ takes 6 different values as we permute the points and

  $$j(\lambda) := 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$$

  is independent of the permutation.
  All $j \in \mathbb{C}$ are obtained as the 4-tuple of points varies.

- Given a nonsingular cubic $C \subset \mathbb{P}^2$ and $P \in C$ there are 4 tangent lines to C passing through P. View them as points of $\mathbb{P}^1$, and compute their $j(\lambda)$. Then $j(\lambda)$ is independent of P. Call it $j(C)$.

  For example if C is in Hesse normal form

  $$X_0^3 + X_1^3 + X_2^3 + 6\alpha X_0 X_1 X_2 = 0$$

  then $1 + 8\alpha^3 \neq 0$ and $j(C) = \frac{64(\alpha - \alpha^4)^3}{(1+8\alpha^3)^3}$.
- For every $j \in \mathbb{C}$ there exists a cubic C such that $j = j(C)$.
- (Salmon) Two cubics C, C' are isomorphic if and only if $j(C) = j(C')$.

It is difficult to distinguish which parameters are moduli.

For example consider the following linear pencil of plane quartics:

$$\lambda F_4(X_0, X_1, X_2) + \mu(X_0^4 + X_1^4 + X_2^4) = 0, \quad (\lambda, \mu) \in \mathbb{P}^1$$

where $F_4(X_0, X_1, X_2)$ is a general quartic homogeneous polynomial.

The two quartics $F_4(X_0, X_1, X_2) = 0$ and $X_0^4 + X_1^4 + X_2^4 = 0$ are non isomorphic because F_4 has 24 ordinary flexes and the other quartic has 12 hyperflexes.

How can we be sure that the pencil depends on one modulus?

Families

A family *of projective nonsingular curves of genus g* is a projective smooth morphism:

$$f : \mathcal{C} \longrightarrow B$$

whose fibres are nonsingular curves of genus g. Recall that the genus of a nonsingular curve C is

$$g(C) := \dim(H^1(C, \mathcal{O}_C)) = \dim(H^0(C, \Omega^1_C))$$

A family of deformations of a given curve (always projective and nonsingular) C is a pullback diagram:

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d, "f"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "b"] & B \end{tikzcd} \tag{3}$$

where f is a family of proj. n.s. curves of genus g. This means that an isomorphism $C \cong \mathcal{C}(b)$ is given. Can replace smooth curves by possibly singular ones, but then have to require that the family is flat.

An isomorphism between two families of curves $f : \mathcal{C} \to B$ and $\varphi : \mathcal{D} \to B$ is just a B-isomorphism:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "\Phi"] \arrow[dr, "f"'] & & \mathcal{D} \arrow[dl, "\varphi"] \\
& B &
\end{tikzcd}$$

An isomorphism between two families of deformations of C, say (3) and

$$\begin{tikzcd}
C \arrow[r, hook] \arrow[d] & \mathcal{D} \arrow[d, "\varphi"] \\
\mathrm{Spec}(\mathbb{C}) \arrow[r, "b"] & B
\end{tikzcd}$$

is an isomorphism between the families f and φ which commutes with the identifications of C with $\mathcal{C}(b)$ and with $\mathcal{D}(b)$.

A family of curves embedded in a projective variety X is a commutative diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, "j"] \arrow[d, "f"'] & B \times X \arrow[dl, "\pi"] \\ B & \end{tikzcd}$$

where f is a family of projective curve of genus g and π is the projection.
Most important case: $X = \mathbb{P}^r$. One may include the case $r = 1$ by replacing the inclusion j by a finite flat morphism. In this case for each closed point $b \in B$ the fibre $j(b) : \mathcal{C}(b) \to \mathbb{P}^1$ will be a ramified cover of constant degree.

If B is irreducible, or just equidimensional, then $\dim(B)$ is defined to be the number of parameters of the family f.

For example, the pencil of plane quartics considered before defines a family of curves embedded in $\mathbb{P}^2$:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook] \arrow[d] & \mathbb{P}^1 \times \mathbb{P}^2 \arrow[dl] \\
\mathbb{P}^1 &
\end{tikzcd}$$

parametrized by $\mathbb{P}^1$, where $\mathcal{C}$ is defined by the bihomogeneous equation of the pencil.

Moduli functors

We expect a moduli space of curves to be an algebraic $\mathbb{C}$-scheme whose closed points are in 1–1 correspondence with the set $\{\text{genus } g \text{ curves}\}$ of isomorphism classes of curves of a given genus g.
In the case $g = 1$ the affine line $\mathbb{A}^1$ does the job.
But where should its structure of scheme come from?

We expect that it reflects somehow the structure of the set of all families of curves of genus g.

Every scheme X is identified with its representable contravariant functor of points $h_X(-) = \mathrm{Mor}(-, X)$ and it is this functor that gives X the scheme structure.
So we must look for a functor on the first place, and it must be a functor related with families of curves of genus g. Here is one.

Setting

$$\mathcal{M}_g(B) = \{\text{families } \mathcal{C} \to B \text{ of curves of genus } g\}/\text{isomorphism}$$

we obtain a contravariant functor

$$\mathcal{M}_g : (\text{Schemes}) \to (\text{Sets})$$

called the moduli functor of nonsingular curves of genus g.

The optimistic expectation is that $\mathcal{M}_g$ is representable, i.e. that there is a scheme M_g equipped with a *universal family* $\pi : \mathcal{X} \to M_g$ of curves of genus g.
”Universal” means that every other family $f : \mathcal{C} \to B$ of curves of genus g is obtained by pulling back π via a unique morphism $B \to M_g$.
The pair (M_g, π) would represent the functor $\mathcal{M}_g$. In other words it would imply the existence of an isomorphism of functors

$$\mathcal{M}_g \cong h_{M_g}$$

and it would be fair to call such M_g *the moduli space* (or moduli scheme) of curves of genus g. Actually its name would the fine moduli space.

The situation is not that simple though.
Such a family does not exist and this is due to the fact that curves may have automorphisms.

Example There is only one isomorphism class of curves of genus zero, namely $[\mathbb{P}^1]$. So if $\mathcal{M}_0$ were representable the universal family would just be $\mathbb{P}^1 \to \mathrm{Spec}(\mathbb{C})$. But this contradicts the existence of non-trivial ruled surfaces $S \to B$.

Toy example Consider the scheme $X = \mathrm{Spec}(\mathbb{C}[Z]/(Z^2 - 1))$. It consists of two distinct reduced points. It is clearly rigid. So its fine moduli space, if it exists, must be a point, and the universal family must be $X \longrightarrow \mathrm{Spec}(\mathbb{C})$. But this cannot be because there are non-trivial families of deformations of X. For example

$$\mathrm{Spec}(\mathbb{C}[Z, t, t^{-1}]/(Z^2 - t)) \longrightarrow \mathrm{Spec}(\mathbb{C}[t, t^{-1}])$$

No panic: we still are on the right track because any reasonable structure on $\{$genus g curves$\}$ must be somehow compatible with the moduli functor. All we have to do is to weaken somehow the condition that there is a universal family. There are several ways to do this. The first one is via the notion of *coarse moduli space.*

The coarse moduli space of curves

The following definition is due to Mumford.

The coarse moduli space of curves of genus g is a scheme M_g such that:

- There is a morphism of functors $\mathcal{M}_g \to h_{M_g}$ which induces a bijection

$$\mathcal{M}_g(\mathbb{C}) \cong M_g(\mathbb{C}) = \{\mathbb{C}\text{-rational points of } M_g\}$$

- If N is another scheme such that there is a morphism of functors $\mathcal{M}_g \to h_N$ inducing a bijection

$$\mathcal{M}_g(\mathbb{C}) \cong N(\mathbb{C})$$

then there is a unique morphism $M_g \to N$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{M}_g \arrow[r] \arrow[rr, bend left] & h_{M_g} \arrow[r] & h_N
\end{tikzcd}$$

The definition implies that:

- The closed points of M_g are in 1–1 correspondence with the isomorphism classes of (nonsingular projective) curves of genus g.
- for every family $f : \mathcal{C} \to B$ of curves of genus g the set theoretic map

$$(b \in B) \mapsto [f^{-1}(b)] \in M_g$$

  defines a morphism $\mu_f : B \to M_g$. (this is the universal property of M_g).

It is easy to prove that, if it exists, M_g is unique up to isomorphism. In that case we say that $\mathcal{M}_g$ is coarsely represented by M_g. The following is a highly non-trivial result.

Theorem

- *(Mumford) M_g exists and is a quasi-projective normal algebraic scheme.*
- *(Deligne-Mumford, Fulton) M_g is irreducible.*

Moduli have to be interpreted as local parameters on M_g around a given point $[C]$ and the number of moduli on which an abstract curve depends as the dimension of M_g.

Now it is clear, at least theoretically, *how to distinguish moduli among parameters:*

A family of curves $f : \mathcal{C} \to B$, with B an irreducible scheme, depends on $\dim(B)$ parameters and on $\dim(\mu_f(B))$ moduli.

For example in the trivial family

$$f : C \times B \longrightarrow B$$

all fibres are isomorphic to C and therefore $\mu_f(B) = \{[C]\}$: thus the number of moduli of this family is zero. More generally an isotrivial family is a family such that all fibres are isomorphic: it has no moduli. There exist families which are isotrivial but non-trivial.

A simple example of this phenomenon is given by any non-trivial *ruled surface.*

On the opposite side, an effectively parametrized family is one which has $\dim(B)$ moduli and such that μ_f is finite onto its image. This means that the fibre over any point of B is isomorphic to only finitely many others.

Example: Every 1-parameter family which contains two non-isomorphic fibres is effectively parametrized.

In particular the pencil of plane quartics considered before depends on 1 modulus.

A family $f : \mathcal{C} \to B$ has general moduli if $\mu_f : B \to M_g$ is dominant.

An effectively parametrized family such that μ_f is surjective (in particular having general moduli) is called a modular family. Modular families are important in the effective construction of M_g.

Variants: moduli of pointed curves

Given $g \geq 0$ and $n \geq 1$ a useful variant of M_g is the coarse moduli space $M_{g,n}$ of n-pointed curves of genus g.

It parametrizes pairs $(C; p_1, \dots, p_n)$ consisting of a curve C of genus g and an ordered n-tuple $(p_1, \dots, p_n)$ of distinct points of C.

The corresponding moduli functor is

$$\mathcal{M}_{g,n}(B) = \{(f : \mathcal{C} \to B, \sigma_1, \dots, \sigma_m)\} / \text{isomorphism}$$

where $\sigma_1, \dots, \sigma_n : B \to \mathcal{C}$ are disjoint sections of $f : \mathcal{C} \to B$., and the notion of isomorphism is the obvious one.

Riemann's count

Riemann was able to count the number of moduli of curves of genus g, i.e. $\dim(M_g)$, by exhibiting a family of curves of genus g with general moduli in the following way. Assume $g \geq 4$.

Consider the family of all ramified covers of $\mathbb{P}^1$ of genus g and of a fixed degree n such that

$$\frac{g+2}{2} \leq n \leq g-1$$

We can represent it as

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "j"] \arrow[d, "f"'] & B \times \mathbb{P}^1 \arrow[dl, "\pi"] \\
B &
\end{tikzcd}$$

where B is a certain irreducible scheme. Riemann existence theorem implies that, associating to a cover the set of its branch points we obtain a *finite* morphism $r : B \to (\mathbb{P}^1)^{2(n+g-1)}$.

Therefore $\dim(B) = 2(n+g-1)$.

Consider $\mu_f : B \to M_g$. We have the following facts:

- each curve of genus g can be expressed as a ramified cover of $\mathbb{P}^1$ defined by a line bundle of degree n provided $n \geq \frac{g+2}{2}$. Therefore μ_f is dominant.
- Composing a cover $f : C \to \mathbb{P}^1$ with a non-trivial automorphism $\alpha : \mathbb{P}^1 \to \mathbb{P}^1$ we obtain another cover $\alpha \cdot f : C \to \mathbb{P}^1$ defined on the same curve by the same line bundle L.
- In the range $\frac{g+2}{2} \leq n \leq g - 1$ the line bundles L of degree n with two sections on a given curve C depend on $2n - 2 - g$ parameters.

Therefore:

the general fibre of μ_f has dimension

$$\dim(\mathrm{PGL}(2)) + 2n - 2 - g = 2n + 1 - g$$

Then we conclude that:

$$\begin{aligned}\dim(M_g) &= \dim(\mathrm{Im}(\mu_f))\\ &= \dim(B) - (2n + 1 - g)\\ &= 2(n + g - 1) - (2n + 1 - g)\\ &= 3g - 3\end{aligned}$$

This computation depends on several implicit assumptions but is essentially correct.
If $g = 2, 3$ one can take $n = 2, 3$ resp. and get the same result by a similar computation.
For $g = 0, 1$ we get

$$\dim(M_0) = 0, \quad \dim(M_1) = 1$$

The previous computation is an example of parameter counting, a method that can be applied in several situations and is useful in computing the dimension of various loci in M_g.
For such computations the universal property of M_g is used. There is a better way to perform them and it is by means of *deformation theory* (see later).

Stable curves

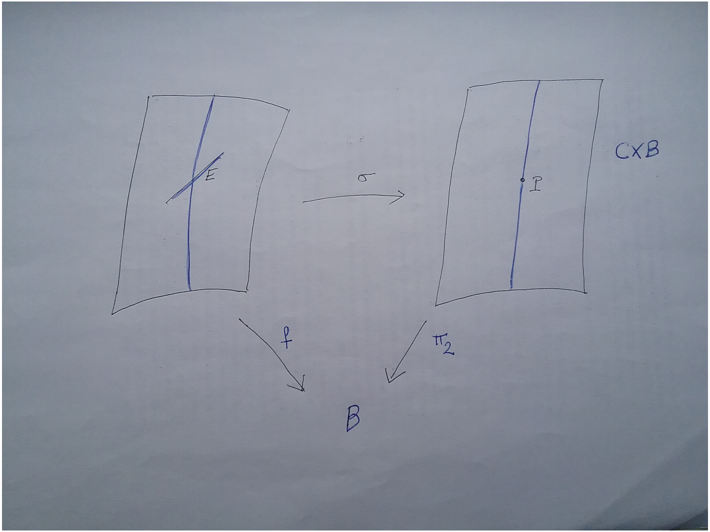
The moduli space M_g is quasi-projective but for all positive g it is not projective. The reason is that curves varying in a family may become singular. In that case one speaks of a degenerating family of curves.

It is therefore natural to consider the functor $\widetilde{\mathcal{M}}_g$, of which $\mathcal{M}_g$ is a subfunctor, defined as follows:

$$\widetilde{\mathcal{M}}_g(B) := \qquad \left\{ \begin{array}{c} \text{isom. classes of flat families} \\ \text{of curves of arithmetic genus } g \end{array} \right\}$$

and ask: is it possible to embed M_g into a projective scheme $\widetilde{M}_g$ which is a coarse moduli scheme for the functor $\widetilde{\mathcal{M}}_g$?

No chance. Consider a nonsingular curve C and a parameter nonsingular curve B. Let $P \in C \times B$ and consider the following family f:



But there is a nice solution if we modify the question by allowing only certain singular curves.

Recall that the arithmetic genus of a reduced projective curve C is

$$p_a(C) := \dim(H^1(C, \mathcal{O}_C))$$

Definition

A stable curve of genus g is a connected reduced curve of arithmetic genus g having at most nodes (i.e. ordinary double points) as singularities and such that every nonsingular rational component meets the rest of the curve in ≥ 3 points.

Define the moduli functor of stable curves of genus g as follows:

$$\overline{\mathcal{M}}_g(B) := \left\{ \begin{array}{c} \text{isom. classes of flat families} \\ \text{of stable curves of genus } g \end{array} \right\}$$

We obviously have:

$$\mathcal{M}_g(B) \subseteq \overline{\mathcal{M}}_g(B) \subseteq \widetilde{\mathcal{M}}_g(B)$$

Theorem (Deligne-Mumford)

There is a projective scheme $\overline{M}_g$ *containing* M_g *and coarsely representing the functor* $\overline{\mathcal{M}}_g$*. The complement* $\overline{M}_g \setminus M_g$ *is a divisor with normal crossings.*

Remark

M_g *is not projective but not affine either. It is known that a priori it may contain projective subvarieties having up to dimension* $g-2$ *(Theorem of Diaz) but the precise bound is not known.*

Other variants of M_g (pointed curves and stable pointed curves) will be introduced later.

The local structure of M_g

It would be very convenient to have the full strength of the universal property.
Unfortunately as we saw, the coarse moduli space M_g does not represent the moduli functor. This implies that not every morphism $B \to M_g$ of schemes is induced by a family of curves $\mathcal{C} \to B$.

We could remedy to this by considering the moduli stack instead of the moduli space. Its definition is not as concise and intuitive as that of coarse moduli space and we do not need it anyway. All we need is a good local description of M_g. Assume $g \geq 2$. Then:

- Each point $[C] \in M_g$ has an open neighborhood of the form V/G where V is nonsingular of dimension $3g - 3$ and G is the finite group $\mathrm{Aut}(C)$.

The nonsingular space V is constructed by means of deformation theory. Roughly:

$$\text{(deformation theory)} \leftrightarrow \begin{pmatrix} \text{small deformations of } C \\ \text{modulo isom. of def.s of } C \end{pmatrix} \tag{4}$$

while

$$\begin{pmatrix} \text{local structure} \\ \text{of } M_g \text{ at } [C] \end{pmatrix} \leftrightarrow \begin{pmatrix} \text{small deformations of } C \\ \text{modulo isom. of def.s} \end{pmatrix} \tag{5}$$

Therefore passing from (4) to (5) one has to quotient by the action of $\mathrm{Aut}(C)$.

Our next step will be to explain some deformation theory.

Deformation Theory

The all point of D.T. is that considering deformations of C modulo isomorphism we obtain a functor very close to being representable. In fact it satisfies a sort of universal property with respect to infinitesimal deformations (it is prorepresentable).
Infinitesimal deformations are deformations parametrized by $\mathrm{Spec}(A)$ where A is an Artin local $\mathbb{C}$-algebra with residue field $\mathbb{C}$.

As a result D.T. constructs a formal deformation parametrized by a complete local $\mathbb{C}$-algebra R as a limit of infinitesimal deformations, which has a universal property with respect to infinitesimal deformations.
The algebra R is the completion at $[C]$ of the local ring of a local space V on which $\mathrm{Aut}(C)$ acts.

The most important informations that can be obtained from D.T. are dimension and nonsingularity of R, i.e. of V at $[C]$.
They are obtained by studying tangent space and obstruction space to the deformation functor.

DIGRESSION:
Suppose that we are given a scheme X and a $\mathbb{C}$-rational point $x \in X$. Let $(A, \mathfrak{m})$ be the local ring of X at x and

$$T_x X = (\mathfrak{m}/\mathfrak{m}^2)^\vee$$

the Zariski tangent space of X at x.

Denote by

$$D := \mathrm{Spec}(\mathbb{C}[\epsilon]) := \mathrm{Spec}(\mathbb{C}[t]/(t^2))$$

Then we can also identify:

$$T_x X = \mathrm{Mor}(D, X)_x = \{v : D \to X : v((\epsilon)) = x\}$$

because to give such a v means to give $A \to \mathbb{C}[\epsilon]$, which is the same as giving $d : \mathfrak{m} \to \mathbb{C}\epsilon$, i.e. a $\mathbb{C}$-linear map $\mathfrak{m}/\mathfrak{m}^2 \to \mathbb{C}$.

Let's translate this simple remark in the setting of deformation theory of C.

A first order deformation of C is a family of deformations of C parametrized by D:

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d, "\varphi"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "(\epsilon)"] & D \end{tikzcd}$$

This deformation defines an element

$$(\kappa(\varphi) : D \to V) \in \mathrm{Mor}(D, V)_{[C]} = T_{[C]}V$$

and therefore, using the universal property, we obtain an identification:

$$\{\text{first order deformations of } C\} = T_{[C]}V$$

Proposition

There is a canonical identification:

$$\{\textit{first order deformations of } C\} = H^1(C, T_C)$$

Proof. (outline) Consider a first order deformation $\varphi : \mathcal{C} \to D$ and an affine open cover of C:

$$\mathcal{U} = \{U_i\} = \{\mathrm{Spec}(R_i)\}$$

then $\mathcal{C} = \bigcup_i \widetilde{U}_i$ where each $\widetilde{U}_i$ is a first order deformation of U_i:

$$\begin{tikzcd}
U_i \arrow[r, hook] \arrow[d] & \widetilde{U}_i \arrow[d, "\varphi"] \\
\mathrm{Spec}(\mathbb{C}) \arrow[r, "b"] & D
\end{tikzcd}$$

Fact: (*rigidity of nonsingular affine schemes*) Every first order deformation of a nonsingular affine scheme is trivial .

Thus

$$\widetilde{U}_i \cong U_i \times D = \mathrm{Spec}(R_i[\epsilon])$$

as deformations, for all i. It follows that to give $\varphi : \mathcal{C} \to D$ is the same as to give patching data

$$\begin{tikzcd}
U_{ij} \times D \arrow[r, "\theta_{ij}"] \arrow[d, hook] & U_{ij} \times D \arrow[d, hook] \\
U_i \times D & U_j \times D
\end{tikzcd}$$

such that $\theta_{ij}\theta_{jk}\theta_{ik}^{-1} = 1_{U_{ijk} \times D}$. The θ_{ij}'s correspond to automorphisms of $R_{ij}[\epsilon]$ inducing the identity modulo ϵ. Each such automorphism is of the form $a + \epsilon b \mapsto a + \epsilon(b + d_{ij}a)$ where $d_{ij} : R_{ij} \to R_{ij}$ is a $\mathbb{C}$-derivation, i.e. $d_{ij} \in \Gamma(U_{ij}, T_C)$.

The cocycle condition translates into the conditions $d_{ij} + d_{jk} - d_{ik} = 0$, so we obtain a 1-cocycle $(d_{ij}) \in \mathcal{Z}^1(\mathcal{U}, T_C)$, which in turn defines an element $\kappa(\varphi) \in H^1(C, T_C)$. One checks that $\kappa(\varphi)$ depends only on the isomorphism class of φ and not on the cover $\mathcal{U}$. Conversely, given $\alpha \in H^1(C, T_C)$ one represents it by a Cech 1-cocycle and constructs a first order deformation $\varphi_\alpha : \mathcal{C} \to D$.
The two operations are inverse to each other, namely:

$$\kappa(\varphi_\alpha) = \alpha, \quad \varphi_{\kappa(\varphi)} \cong \varphi$$

□

The cohomology class $\kappa(\varphi)$ is called Kodaira-Spencer class (KS class) of $\varphi : \mathcal{C} \to D$.

Given a family of deformations of C:

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d, "\varphi"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "b"] & B \end{tikzcd}$$

we have an induced linear map:

$$\kappa : T_b B \longrightarrow H^1(C, T_C)$$

called Kodaira-Spencer map, which associates to a tangent vector $v : D \to B$ at b the KS class $\kappa(\varphi_v)$ of the first order deformation $\varphi_v : \mathcal{C}_v \to D$ of C obtained by pulling back φ to D.

To the family of curves $\varphi : \mathcal{C} \to B$ there is associated the functorial map $\mu_\varphi : B \to M_g$ while to the above family of deformations of C there is associated a map $m : U \to V$ from a neighborhhod U of $b \in B$, and $\kappa = dm_b$. So κ is *almost* the differential of μ_φ:

$$\begin{tikzcd}
 & V \arrow[d] \\
U \arrow[ur, "m"] \arrow[r, "\mu_\varphi"'] & M_g
\end{tikzcd}
\qquad
\begin{tikzcd}
 & H^1 C, T_C) \arrow[d] \\
T_b B \arrow[ur, "\kappa"] \arrow[r, "d\mu_\varphi"'] & T_{[C]} M_g
\end{tikzcd}$$

Nonsingularity and obstructions

The nonsingularity of V at $[C]$ corresponds to the fact that $R = \mathbb{C}[[T_1, \ldots, T_{3g-3}]]$ is a formal power series ring. Using the infinitesimal criterion of formal smoothness this means that given a surjection of local Artin $\mathbb{C}$-algebras $A' \to A = A'/(\epsilon)$ and a commutative diagram of black arrows:

$$\begin{tikzcd}
R \arrow[r] \arrow[dr] & A \\
\mathbb{C} \arrow[u] \arrow[r] & A' \arrow[u]
\end{tikzcd}$$

the arrow $\longrightarrow$ exists.

Nonsingularity and obstructions

The nonsingularity of V at $[C]$ corresponds to the fact that $R = \mathbb{C}[[T_1, \dots, T_{3g-3}]]$ is a formal power series ring. Using the infinitesimal criterion of formal smoothness this means that given a surjection of local Artin $\mathbb{C}$-algebras $A' \to A = A'/(\epsilon)$ and a commutative diagram of black arrows:

$$\begin{tikzcd}
R \arrow[r] \arrow[dr, blue] & A \\
\mathbb{C} \arrow[u] \arrow[r] & A' \arrow[u]
\end{tikzcd}$$

the arrow $\longrightarrow$ exists.

This diagram corresponds to a diagram of infinitesimal deformations

$$\begin{tikzcd}
C \arrow[r, hook] \arrow[d] & \mathcal{C}_A \arrow[r, hook, blue] \arrow[d] & \color{blue}{\mathcal{C}_{A'}} \arrow[d, blue] \\
\mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(A) \arrow[r, hook] & \mathrm{Spec}(A')
\end{tikzcd}$$

The condition of existence of $\mathcal{C}_{A'}$ translates into a 2-cocycle condition with coefficients in T_C. This obstruction to the existence of $\mathcal{C}_{A'}$ is an element of $H^2(C, T_C)$. Since this group vanishes $\mathcal{C}_{A'}$ exists and therefore R is smooth.
As a consequence we obtain again:

Corollary $\dim(M_g) = \dim(V) = h^1(C, T_C) = 3g - 3$
if $g \geq 2$.

The Hilbert scheme

Fix a polynomial $p(t) \in \mathbb{Q}[t]$ and define a contravariant functor:

$$Hilb^r_{p(t)} : (\text{Schemes}) \longrightarrow (\text{Sets})$$

setting

$$Hilb^r_{p(t)}(B) = \left\{ \begin{array}{c} \text{families of closed subschemes of } \mathbb{P}^r \\ \text{param. by } B \text{ and with Hilbert polyn. } p(t) \end{array} \right\}$$

This is the Hilbert functor for the polynomial $p(t)$.
When $p(t) = dt + 1 - g$, for integers d, g then we write $Hilb^r_{d,g}$.

Theorem (Grothendieck) *For every $r \geq 2$ and $p(t)$ there is a projective scheme* $\mathrm{Hilb}^r_{p(t)}$ *and a family*

$$\begin{tikzcd}
\mathcal{X} \arrow[r, hook] \arrow[d] & \mathrm{Hilb}^r_{p(t)} \times \mathbb{P}^r \arrow[dl] \\
\mathrm{Hilb}^r_{p(t)} &
\end{tikzcd}$$

which is universal for the functor $Hilb^r_{p(t)}$. In particular $Hilb^r_{p(t)}$ is representable.

$\mathrm{Hilb}^r_{p(t)}$ is called Hilbert scheme of $\mathbb{P}^r$ relative to the Hilbert polynomial $p(t)$.
It is a very complicated object. Its local properties at a point $[X \subset \mathbb{P}^r]$ depend only on the geometry of the embedding $X \subset \mathbb{P}^r$.

Theorem *Let $X \subset \mathbb{P}^r$ be a local complete intersection with Hilbert polynomial $p(t)$. Let $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^r}$ be its ideal sheaf and $N = N_{X/\mathbb{P}^r} := Hom(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$ its normal bundle. Then:*

- $H^0(X, N)$ is the Zariski tangent space to $\mathrm{Hilb}^r_{p(t)}$ at $[X]$.
- $h^0(X, N) - h^1(X, N) \leq \dim_{[X]}(\mathrm{Hilb}^r_{p(t)}) \leq h^0(X, N)$.
- If $H^1(X, N) = 0$ then $\mathrm{Hilb}^r_{p(t)}$ is nonsingular of dimension $h^0(X, N)$ at $[X]$.

Special case: $C \subset \mathbb{P}^r$ is a nonsingular curve of degree d and genus g. Then $p(t) = dt + 1 - g$ and we write $\mathrm{Hilb}^r_{d,g}$ instead of $\mathrm{Hilb}^r_{p(t)}$. Then:

$$h^0(C, N) - h^1(C, N) = \chi(C, N) = (r + 1)d + (r - 3)(1 - g)$$

Examples

- Nonsingular plane curves of degree d. Their genus is $g = \binom{d-1}{2}$ and $N = \mathcal{O}_C(d)$. Then $H^1(C, N) = 0$ and

$$h^0(C, N) = 3d + g - 1 = \frac{d(d+3)}{2} = \binom{d+2}{2} - 1$$

- Nonsingular curves in $\mathbb{P}^3$ of degree d and genus g. In this case $\chi(N) = 4d$ does not depend on g.
  $\mathrm{Hilb}^3_{d,g}$ can be singular and/or of dimension $> 4d$.
- Nonsingular curves in $\mathbb{P}^r$, $r \geq 4$ of degree d and genus g. If g is large with respect to d then $\chi(N) < 0$. This has interesting modular interpretation.

The Kodaira-Spencer map of the Hilbert scheme

Suppose given $C \subset \mathbb{P}^r$ nonsingular of degree d and genus g. It corresponds to a point $[C] \in \mathrm{Hilb}^r_{d,g}$ and therefore there is a KS map:

$$\kappa_C : H^0(C, N_{C/\mathbb{P}^r}) \longrightarrow H^1(C, T_C)$$

Proposition

κ_C is the coboundary map of the normal sequence

$$\begin{tikzcd} 0 \arrow[r] & T_C \arrow[r] & T_{\mathbb{P}^r|C} \arrow[r] & N_{C/\mathbb{P}^r} \arrow[r] & 0 \end{tikzcd}$$

Proof. by easy diagram chasing. ☐

General curves

Definition: A general curve of genus g is a curve parametrized by a general point of M_g.

If we want to find general curves we must produce families of curves with general moduli, i.e. families $f : \mathcal{C} \to B$ such that $\mu_f : B \to M_g$ is dominant.

In order to check if this is the case for a given family it suffices to produce a nonsingular point $b \in B$ such that the KS map $\kappa_b : T_b B \to H^1(\mathcal{C}(b), T_{\mathcal{C}(b)})$ is surjective.

Let's consider $C \subset \mathbb{P}^r$ and suppose we want to check whether the universal family over $\mathrm{Hilb}^r_{d,g}$ has general moduli around $[C]$. A sufficient condition would be that $H^1(C, T_{\mathbb{P}^r|C}) = 0$. We can use the restricted Euler sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_C \arrow[r] & H^0(L)^\vee \otimes L \arrow[r] & T_{\mathbb{P}^r|C} \arrow[r] & 0 \end{tikzcd}$$

where $L = \mathcal{O}_C(1)$. Then $H^1(C, T_{\mathbb{P}^r|C}) = 0$ if and only if the map:

$$H^1(\mathcal{O}_C) \longrightarrow H^0(L)^\vee \otimes H^1(L)$$

is surjective, if and only if its dual:

$$\mu_0(L) : H^0(L) \otimes H^0(\omega_C L^{-1}) \longrightarrow H^0(\omega_C)$$

is injective. $\mu_0(L)$ is called the Petri map of L.

Definition

A curve C is called Petri general if the Petri map $\mu_0(L)$ is injective for all invertible sheaves $L \in \mathrm{Pic}(C)$.

Petri, a student of M. Noether, in a footnote to a paper of 1923, stated as a fact what has been subsequently considered as

Petri's conjecture: For every g a general curve of genus g is Petri general.

By semicontinuity Petri's conjecture is equivalent to the existence of just one Petri general curve of genus g for each g. This is easy to do for small values of g, but it becomes increasingly difficult as g grows.

According to the conjecture the Petri general curves should be the most natural ones available in nature. But in fact this is not the case.

For example: nonsingular plane curves of degree $d \geq 5$ are not Petri general, nor are most complete intersections in $\mathbb{P}^r$.
In fact a simple remark shows that $H^0(\omega_C L^{-2}) \subset \ker(\mu_0(L))$.
Take $C \subset \mathbb{P}^2$ of degree d. Then $\omega_C = \mathcal{O}(d-3)$ and therefore

$$H^0(\omega_C L^{-2}) = H^1(\mathcal{O}_C(2))^\vee \neq 0$$

if $d \geq 5$. A similar remark holds for complete intersections of multidegree $(d_1, \ldots, d_{r-1})$ such that $\sum d_j \geq r+3$.

It is very difficult to produce explicit examples of Petri general curves. So the challenge of Petri's conjecture, if true, is to change our naive idea of a general curve.

The conjecture is in fact true. It has been proved for the first time by Gieseker (1982), and subsequently it has been given simpler proofs by Eisenbud-Harris (1983) and by Lazarsfeld (1986). Special cases of the conjecture had been proved before by Arbarello-Cornalba (1981).

The Petri map is a central object in curve theory. Not only its kernel but also its cokernel is very important.

If $\deg(L) = d$ and $h^0(L) = r + 1$ then the expected corank of $\mu_0(L)$ is

$$\rho(g, r, n) := g - (r + 1)(g - d + r)$$

This number is called Brill-Noether number relative to g, r, d.

The name tells us that this number was introduced by Brill and Noether much before Petri conceived the maps $\mu_0(L)$.

Brill-Noether Theory

The Picard group of isomorphism classes of invertible sheaves on a nonsingular curve C decomposes as

$$\mathrm{Pic}(C) = \coprod_{d \in \mathbb{Z}} \mathrm{Pic}(C)_d$$

where $\mathrm{Pic}(C)_d$ consists of the sheaves of degree d. The subgroup $\mathrm{Pic}(C)_0$ is an abelian variety, called the jacobian variety of C, also denoted by $J(C)$. The geometric structure of $\mathrm{Pic}(C)_0$ is transferred on each $\mathrm{Pic}(C)_d$ using the fact that the degree is a group homomorphism:

$$\mathrm{Pic}(C) \longrightarrow \mathbb{Z}$$

The group structure on $\mathrm{Pic}_0(C)$ trivializes the tangent bundle, hence all the tangent spaces to $\mathrm{Pic}(C)$ are canonically isomorphic.

Theorem For any $L \in \mathrm{Pic}(C)$ we have

$$T_L \mathrm{Pic}(C) = H^1(C, \mathcal{O}_C)$$

canonically.

Proof. The transcendental theory tells us that

$$J(C) = \frac{H^1(C, \mathcal{O}_C)}{H^1(C, \mathbb{Z})}$$

and the theorem follows from this fact. For our purposes it is convenient to have a deformation theoretic proof.

$\mathrm{Pic}(C)$ represents the Picard functor:

$$Pic_C(B) = \{\mathcal{L} \text{ on } C \times B\}/\{\pi_B^* A,\ A \in \mathrm{Pic}(B)\}$$

Therefore $T_L \mathrm{Pic}(C)$ is identified with the set of first order deformations of L.

Suppose L represented by a 1-cocycle $\{f_{ij}\} \in \mathcal{Z}^1(\mathcal{U}, \mathcal{O}_C^*)$ with respect to an affine covering $\mathcal{U} = \{U_i\}$. Then we can represent a first order deformation $\mathcal{L}$ of L by transition functions

$$F_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_{C \times D}^*) = (\Gamma(U_i \cap U_j, \mathcal{O}_C^* + \epsilon \mathcal{O}_C)$$

which restrict to the f_{ij}'s mod ϵ. Therefore they are of the form:

$$F_{ij} = f_{ij}(1 + \epsilon g_{ij})$$

with $g_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_C)$. The cocycle condition $F_{ij} F_{jk} = F_{ik}$ translates into $g_{ij} + g_{jk} = g_{ik}$ and therefore $\{g_{ij}\} \in \mathcal{Z}^1(\mathcal{U}, \mathcal{O}_C)$. □

The key objects of the theory are the BN schemes, set theoretically defined by:

$$W_d^r(C) = \{L \in \mathrm{Pic}_d(C) : h^0(L) \geq r+1\}$$

They have a scheme theoretic definition, as degeneracy loci of certain maps of vector bundles. Infinitesimally, we can give the following description.

Let $L \in W^r_d(C)$ and $\theta \in H^1(C, \mathcal{O}_C) = T_L\mathrm{Pic}(C)$. The condition $\theta \in T_L W^r_d(C)$ is that all sections of L can be lifted to sections of the deformation $\mathcal{L}$ of L defined by θ. Consider the dual of the Petri map:

$$\mu_0(L)^\vee : H^1(C, \mathcal{O}_C) \longrightarrow \mathrm{Hom}(H^0(L), H^1(L))$$

and let $t = \mu_0(L)^\vee(\theta) : H^0(L) \to H^1(L)$.
An easy cocycle computation shows that $\sigma \in H^0(L)$ lifts to a section of $\mathcal{L}$ if and only if $\sigma \in \ker(t)$. Therefore:

$$T_L W^r_d(C) = \ker(\mu_0^\vee(L))$$

In particular:

$$\rho(g, r, d) \leq \dim_L(W^r_d(C)) \leq \dim \ker(\mu_0^\vee(L))$$

The first inequality is a consequence of the definition of $W^r_d(C)$.
Then:

Theorem

$W^r_d(C)$ is nonsingular of dimension $\rho(g, r, d)$ at L if and only if $\mu_0(L)$ is injective.

As a consequence we obtain:

Corollary

Assume that C is Petri general. Then $W^r_d(C) = \emptyset$ if $\rho(g, r, d) < 0$. If $W^r_d(C) \neq \emptyset$ then it has dimension $\rho(g, r, d)$ and its singular locus is $W^{r+1}_d(C)$.

Note: The statement of the corollary for general curves has been proved by Griffiths-Harris in 1980.

In fact we know more:

Theorem

- *(Kleiman-Laksov (1972), Kempf (1972)) For every curve C of genus g*
$$W^r_d(C) \neq \emptyset \quad \textit{if } \rho(g, r, d) \geq 0$$
- *(Fulton-Lazarsfeld (1981)) If $\rho(g, r, d) > 0$ then $W^r_d(C)$ is connected and it is irreducible if C is general.*

The condition $\rho(g, r, d) \geq 0$ is:

$d \geq \frac{1}{2}g + 1$ if $r = 1$,

$d \geq \frac{2}{3}g + 2$ if $r = 2$, etc.

The following table summarizes and explains the above discussion.

General vs Petri general

To produce a general curve of genus g is the most difficult and elusive part of the theory.
It can be done with different degrees of accuracy.

- One can apply the general results of BN theory to conclude that whenever $\rho(g, r, n) \geq 0$ there exists a component of $\mathrm{Hilb}^r_{d,g}$ whose general member has $L = \mathcal{O}(1)$ with injective Petri map, and therefore the corresponding family has general moduli.

This is already a solution. But we want something more concrete.

We can point to two different directions.

- To produce families $f : \mathcal{C} \longrightarrow B$ with general moduli whose parameter space has specific properties.
- To produce Petri general curves in an effective way.

In the first case we will obtain a dominant morphism $\mu_f : B \longrightarrow M_g$. If B has low Kodaira dimension then also M_g does. In particular if B is rational then M_g will be unirational. The search for such a family has a long history and has motivated a large amount of work around M_g.

The second direction is more recent and has proved to be quite important for a better understanding of the geometry of M_g.

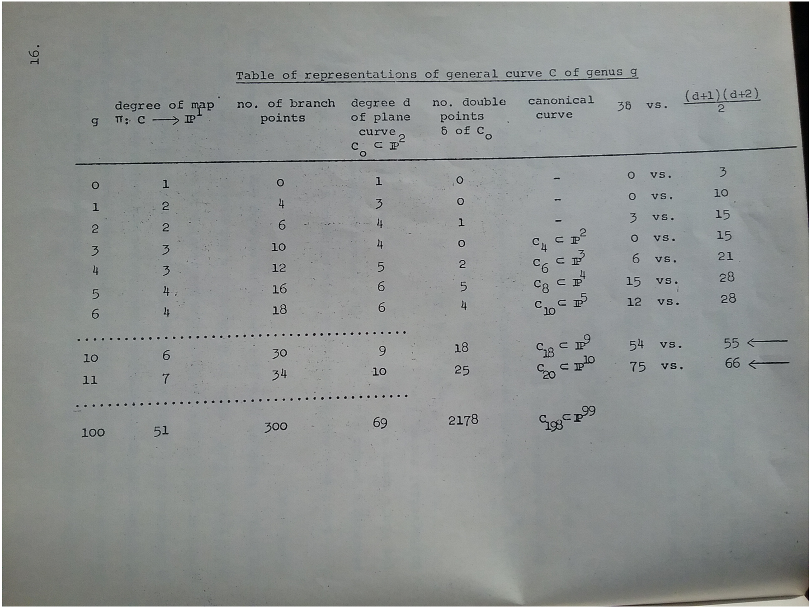
Unirationality or non-unirationality?

A canonical curve of genus 3 is just a nonsingular plane quartic: it moves in a linear system, so it can be parametrized by *free* parameters. Since the family of canonical curves has general moduli this means that we can dominate M_3 by a rational variety. So M_3 is unirational.
A similar remark can be made for genus 4 and 5 since canonical curves of genus 4 and 5 are complete intersections in $\mathbb{P}^3$ (resp. $\mathbb{P}^4$).

One may ask whether an analogous statement is true for higher values of g, namely whether it is possible to produce a family of curves with general moduli and parametrized by a rational variety, say an open subset of a projective space.

To my knowledge M. Noether was the first to ask such a question. He proved (1885) the unirationality of M_g up to genus 7.

Subsequently (1915) Severi extended the result up to genus 10 and conjectured the unirationality of M_g for all g.
The proof given by Severi is quite simple and can be easily understood by looking at the following table.



16.

Table of representations of general curve C of genus g

g	degree of map $\pi: C \longrightarrow \mathbb{P}^1$	no. of branch points	degree d of plane curve $C_o \subset \mathbb{P}^2$	no. double points δ of C_o	canonical curve	3δ vs.	$\frac{(d+1)(d+2)}{2}$
0	1	0	1	0	-	0 vs.	3
1	2	4	3	0	-	0 vs.	10
2	2	6	4	1	-	3 vs.	15
3	3	10	4	0	$C_4 \subset \mathbb{P}^2$	0 vs.	15
4	3	12	5	2	$C_6 \subset \mathbb{P}^3$	6 vs.	21
5	4	16	6	5	$C_8 \subset \mathbb{P}^4$	15 vs.	28
6	4	18	6	4	$C_{10} \subset \mathbb{P}^5$	12 vs.	28
....	....	....	....				
10	6	30	9	18	$C_{18} \subset \mathbb{P}^9$	54 vs.	55 ←
11	7	34	10	25	$C_{20} \subset \mathbb{P}^{10}$	75 vs.	66 ←
....	....	....	....				
100	51	300	69	2178	$C_{198} \subset \mathbb{P}^{99}$		

Table from Mumford's Curves and their jacobians

Outline:

If $g \leq 10$ it is possible to realize a general curve of genus g as a plane curve of degree d with δ singular points in such a way that

$$3\delta \leq \frac{d(d+3)}{2}$$

This implies, modulo a careful argument, that we can assign the singular points of such a curve in general position. Then the parameter space of the family $f : \mathcal{C} \to B$ of plane curves of degree d and genus g is fibered over $(\mathbb{P}^2)^{(\delta)}$ with fibres linear systems, and therefore B is rational.

State of the art about Kodaira dimension of M_g

genus	K-dim	credit
$1, 2, 3, 4, 6$	rational	Weierstrass-Salmon (1), Igusa (2) Katsylo (3), Shepherd-Barron (4,6)
11	uniruled	Mori-Mukai
≤ 14	unirational	Noether (≤ 7), Severi (≤ 10), Sernesi (12), Chang-Ran ($11, 13$), Verra (14)
15	rat.lly connected	Chang-Ran ($-\infty$), Bruno-Verra
16	uniruled	Chang-Ran ($-\infty$), Farkas
$22, \geq 24$	gen. type	Farkas (22), Harris-Mumford (odd), Eisenbud-Harris (even)
23	≥ 2	Farkas

Petri general curves

It is very difficult to find explicit examples of Petri general curves. The most natural examples of curves given in nature (e.g. nonsingular complete intersections) are not Petri general. Their existence has been proved originally by degeneration (Gieseker 1982). The following has been a breakthrough:

Theorem (Lazarsfeld (1986))

If S *is a K3 surface with* $\mathrm{Pic}(S) = \mathbb{Z}[H]$ *then a general curve* $C \in |H|$ *is Petri general.*

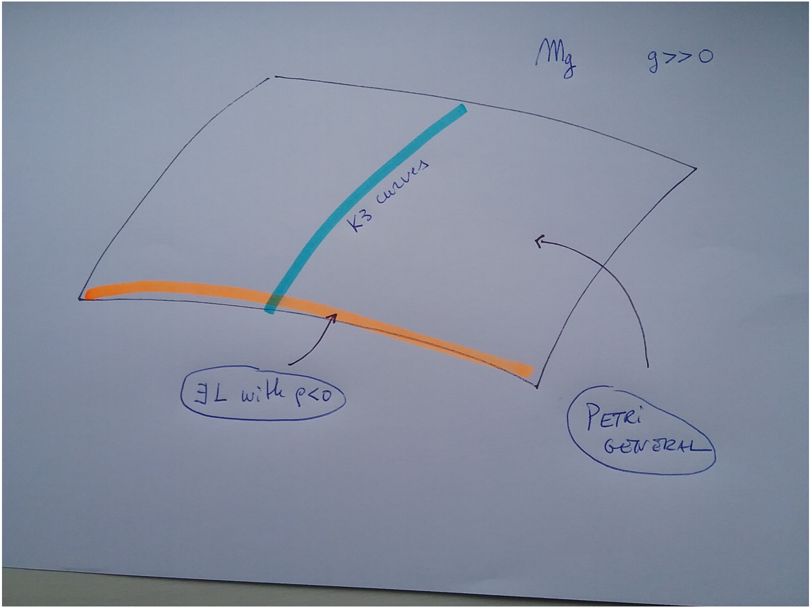
Therefore from the point of view of BN theory complete intersections are special while curves on K3 surfaces are “general”.

But being “general” in the above sense does not mean that they are general in the modular sense.

Count of parameters:

- Pairs (S, H) depend on 19 moduli.
- The linear system $|H|$ on a given polarized S has dimension $g = g(H)$.
- Therefore the locus of curves in M_g that can be embedded in a K3 surface depend on $\leq g + 19$ moduli.
- If $g \geq 12$ then $3g - 3 > g + 19$ and therefore the general curve of genus $g \geq 12$ cannot be embedded in a K3 surface.

Therefore K3 curves of sufficiently high genus are not general curves, since they fill a proper locus in M_g.
The picture is the following.


$\mathcal{M}_g$ $g >> 0$
K3 curves
∃L with $\rho < 0$
PETRI GENERAL

Remarks

- If $\rho(g, r, d) < 0$ the curves of genus g with a line bundle L of degree d such that $h^0(L) \geq r + 1$ are contained in a proper locus whose closure is called $M^r_{g,d} \subset M_g$.
  If $\rho(g, r, d) = -1$ then $M^r_{g,d}$ is a divisor. These divisors have proved to be important in the study of the birational properties of M_g. Refer to Farkas' lectures for details.
- Petri general curves of genus $g \geq 12$ contained in a K3 surface have been characterized recently by a cohomological condition. The condition is that the so-called Wahl map

  $$\bigwedge^2 H^0(\omega_C) \longrightarrow H^0(\omega^3)$$

  is not surjective (Arbarello-Bruno-S.).