Higher-Dimensional Varieties

Lecture notes for the
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on
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Olivier Debarre

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Contents

Abstract

The aim of this course is to provide an introduction to Mori's Minimal Model Program (MMP) on smooth projective varieties. We review classical results on Weil and Cartier divisors and define ample and nef divisors. We explain how an asymptotic Riemann–Roch theorem gives a general definition for the intersection of Cartier divisors. We also go through the construction of the moduli space of morphisms from a fixed curve to a fixed smooth variety, define free curves and uniruled varieties, and state Mori's bend-and-break lemmas. We finish with a proof of Mori's cone theorem for smooth projective varieties and explain the basic steps of the MMP.

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Introduction

Mori's Minimal Model Program is a classification program: given a (smooth) projective variety X, the aim is to find a "simple" birational model of X, ideally one whose canonical bundle is nef (although this is not possible for varieties covered by rational curves). Mori's original approach gave a prominent role to rational curves. Although it is not sufficient to complete his program (essentially because it cannot deal with the necessary evil of singular varieties), it contains beautiful geometric results which have their own interest and which are also much more accessible than the latest developments of the MMP.

Assuming that the reader is familiar with the basics of algebraic geometry (e.g., the contents of the book [H]), we present in these notes the necessary material (and a bit more) to understand Mori's cone theorem.

In Chapter 1, we review Weil and Cartier divisors and linear equivalence (this is covered in [H]). We explain the relation between Cartier divisors and invertible sheaves and define the Picard group. We define the intersection number between a Cartier divisor and a curve. This is a fundamental tool: it defines numerical equivalence, an equivalence relation on the group of Cartier divisors weaker than linear equivalence. The quotient space, the Néron–Severi group, is therefore a quotient of the Picard group and, for proper varieties, it is free abelian of finite rank.

We explain the standard relation between linear systems and rational maps to projective spaces (also covered in [H]). We also explain global generation of coherent sheaves and define ample ($\mathbf{Q}$)-Cartier divisors on a scheme of finite type over a field. We prove Serre's theorems.

After proving a Riemann–Roch theorem on a smooth projective curve, we prove that ample divisors on a smooth projective curve are those of positive degree. Finally, we define, on any projective scheme, nef divisors as those having non-negative intersection number with any curve. We then define, in the Néron–Severi (finite-dimensional real) vector space, the ample, nef, big, effective, and pseudo-effective cones. In order to shorten the exposition, we accept without proof that the sum of an ample divisor and a nef divisor is still ample. This is not very satisfactory since this result is usually obtained as a consequence of the material in the next chapter, but I did not have time to follow the standard (and logical) path.

Chapter 2 is devoted to asymptotic Riemann–Roch theorems: given a Cartier divisor D on a projective scheme X of dimension n, how fast does the dimension $h^0(X, mD)$ of the space of sections of its positive multiples mD grow? It is easier to deal with the Euler characteristic $\chi(X, mD)$ instead, which grows like $am^n/n! +$ lower order terms, where a is an integer which is by definition the self-intersection product (D^n). This is a fast way to define this product, and more generally the intersection product of n Cartier divisors $D_1, \ldots, D_n$ on X. This product turns out to count, when these divisors are effective and meet in only finitely many points, these intersection points with multiplicities. When D is nef, this is also the behavior of $h^0(X, mD)$. We end this chapter by defining and discussing the Kodaira dimension of a Cartier divisor. We define algebraic fibrations and explain what the Iitaka fibration is.

In Chapter 3, we discuss the moduli space of morphisms from a fixed (smooth projective) curve and a projective variety and construct it when the curve is $\mathbf{P}^1$. We explain without proof his local structure. We discuss free rational curves and uniruled varieties (i.e., varieties covered by rational curves). We state (without proof) Mori's bend-and-break lemmas and explain in more details Mori's beautiful proof of the fact that Fano varieties are uniruled (the nice part is the reduction to positive characteristics). In the last section, we explain an extension, based on a classical result of Miyaoka and Mori, of Mori's result to varieties with nef but not numerically trivial anticanonical class.

The fourth and last chapter is devoted to the proof of the cone theorem and its various consequences. The cone theorem describes the structure of the closed convex cone spanned by classes of irreducible curves (the "Mori cone") in the dual of the Néron–Severi (finite-dimensional real) vector space of a smooth projective variety X. It is an elementary consequence of the Miyaoka–Mori theorem mentioned above and uses only elementary geometrical facts on the geometry of closed convex cones in finite-dimensional real vector spaces. We state without proof Kawamata's base-point-free theorem and explain how it allows us to construct, in characteristic 0, contractions of some extremal rays of the Mori cone: these are algebraic fibrations c from X to a projective variety which contract exactly the curves whose class is in the ray.

These fibrations are of 3 types: fiber-type (when all fibers of c have positive dimensions), divisorial (when c is birational with exceptional locus a divisor), small (when c is birational with exceptional locus of codimension ≥ 2). We remark that the image $c(X)$ of the contraction may be singular, but not too singular, except in the case of a small contraction. We end these notes with a one-page description of what the MMP is about and explain what the main problems are.

We prove some, but far from all, the results we state. The bibliography provides a few references where the reader can find more detailed expositions. There are also a few exercises throughout this text.

Chapter 1

Divisors

In this chapter and the rest of these notes, $\mathbf{k}$ is a field and a $\mathbf{k}$-*variety* is an integral scheme of finite type over $\mathbf{k}$.

1.1 Weil and Cartier divisors

In this section, X is a scheme which is for simplicity assumed to be integral.[1]

A *Weil divisor* on X is a (finite) formal linear combination with integral coefficients of integral hypersurfaces in X. Its *support* is the union of the hypersurfaces which appear with non-zero coefficients. We say that the divisor is *effective* if the coefficients are all non-negative.

Assume moreover that X is *normal.* For each integral hypersurface Y of X with generic point η, the integral local ring $\mathscr{O}_{X,\eta}$ has dimension 1 and is regular, hence is a discrete valuation ring with valuation v_Y. For any non-zero rational function f on X, the integer $v_Y(f)$ (valuation of f along Y) is the order of vanishing of f along Y if it is non-negative, and the opposite of the order of the pole of f along Y otherwise. We define the divisor of f as

$$\operatorname{div}(f) = \sum_Y v_Y(f) Y.$$

The rational function f is regular if and only if its divisor is effective ([H, Proposition II.6.3A]).

1.1. Linearly equivalent Weil divisors. Two Weil divisors D and D' on the normal scheme X are *linearly equivalent* if their difference is the divisor of a non-zero rational function on X; we write $D \underset{\text{lin}}{\equiv} D'$. Linear equivalence classes of Weil divisors form a group $\operatorname{Cl}(X)$ (the *divisor class group*) for the addition of divisors.

[1]The definitions can be given for any scheme, but they take a slightly more complicated form.

A *Cartier divisor* is a divisor which can be locally written as the divisor of a non-zero rational function. The formal definition is less enlightening.

Definition 1.2 (Cartier divisors.) *A Cartier divisor on an integral scheme X is a global section of the sheaf $\underline{K(X)}^\times/\mathscr{O}_X^\times$, where $\underline{K(X)}$ is the constant sheaf of rational functions on X.*

In other words, a Cartier divisor is given by a collection of pairs (U_i, f_i), where (U_i) is an affine open cover of X and f_i is a non-zero rational function on U_i such that f_i/f_j is a regular function on $U_i \cap U_j$ that does not vanish.

A Cartier divisor on X is *principal* if it can be defined by a global non-zero rational function on the whole of X.

1.3. Associated Weil divisor. Assume that X is normal. Given a Cartier divisor on X, defined by a collection (U_i, f_i), one can consider the associated Weil divisor $\sum_Y n_Y Y$ on X, where the integer n_Y is the valuation of f_i along $Y \cap U_i$ for any i such that $Y \cap U_i$ is nonempty (it does not depend on the choice of such an i).

A Weil divisor which is linearly equivalent to a Cartier divisor is itself a Cartier divisor.

When X is *locally factorial* (e.g., a smooth variety), i.e., its local rings are unique factorization domains, any hypersurface can be defined locally by one (regular) equation ([H, Proposition II.6.11]),[2] hence any divisor is locally the divisor of a rational function. In other words, there is no distinction between Cartier divisors and Weil divisors.

1.4. Effective Cartier divisors. A Cartier divisor D is *effective* if it can be defined by a collection (U_i, f_i), where f_i is in $\mathscr{O}_X(U_i)$. We write $D \geq 0$. When D is not zero, it defines a subscheme of codimension 1 by the "equation" f_i on each U_i. We still denote it by D.

1.5. Q-divisors. A Weil **Q**-divisor on a scheme X is a (finite) formal linear combination with *rational* coefficients of integral hypersurfaces in X. On a normal scheme X, one says that a **Q**-divisor is **Q**-*Cartier* if some multiple with integral coefficients is a Cartier divisor.

Example 1.6 Let X be the quadric cone defined in $\mathbf{A}^3_\mathbf{k}$ by the equation $xy = z^2$. It is integral and normal. The line L defined by $x = z = 0$ is contained in X; it defines a Weil divisor on X which cannot be defined near the origin by one equation (the ideal (x, z) is not principal in the local ring of X at the origin). It is therefore not a Cartier divisor. However, $2L$ is a principal Cartier divisor, defined by the regular function x, hence L is a **Q**-Cartier divisor. Similarly, the sum of L with the line defined by $y = z = 0$ is also a principal Cartier divisor, defined by the regular function z. So the "components" of a Cartier divisor need not be Cartier.

[2] This is because in a unique factorization domain, prime ideals of height 1 are principal.

1.2 Invertible sheaves

Definition 1.7 (Invertible sheaves) *An invertible sheaf on a scheme X is a locally free $\mathscr{O}_X$-module of rank* 1.

The terminology comes from the fact that the tensor product defines a group structure on the set of locally free sheaves of rank 1 on X, where the inverse of an invertible sheaf $\mathscr{L}$ is $\mathscr{H}om(\mathscr{L}, \mathscr{O}_X)$. This makes the set of isomorphism classes of invertible sheaves on X into an abelian group called the *Picard group* of X, and denoted by $\mathrm{Pic}(X)$. For any $m \in \mathbf{Z}$, it is traditional to write $\mathscr{L}^m$ for the mth (tensor) power of $\mathscr{L}$ (so in particular, $\mathscr{L}^{-1}$ is the dual of $\mathscr{L}$).

Let $\mathscr{L}$ be an invertible sheaf on X. We can cover X with affine open subsets U_i on which $\mathscr{L}$ is trivial and we obtain changes of trivializations, or transition functions

$$g_{ij} \in \mathscr{O}_X^\times(U_i \cap U_j). \tag{1.1}$$

They satisfy the cocycle condition

$$g_{ij}g_{jk}g_{ki} = 1$$

hence define a Čech 1-cocycle for $\mathscr{O}_X^\times$. One checks that this induces an isomorphism

$$\mathrm{Pic}(X) \simeq H^1(X, \mathscr{O}_X^\times). \tag{1.2}$$

For any $m \in \mathbf{Z}$, the invertible sheaf $\mathscr{L}^m$ corresponds to the collection of transition functions $(g_{ij}^m)_{i,j}$.

1.8. Invertible sheaf associated with a Cartier divisor. Given a Cartier divisor D on an integral scheme X, given by a collection (U_i, f_i), one can construct an invertible subsheaf $\mathscr{O}_X(D)$ of $\underline{K(X)}$ by taking the sub-$\mathscr{O}_X$-module generated by $1/f_i$ on U_i. We have

$$\mathscr{O}_X(D_1) \otimes \mathscr{O}_X(D_2) \simeq \mathscr{O}_X(D_1 + D_2).$$

Every invertible subsheaf of $\underline{K(X)}$ is obtained in this way and two Cartier divisors are linearly equivalent if and only if their associated invertible sheaves are isomorphic ([H, Proposition II.6.13]). Since X is integral, every invertible sheaf is a subsheaf of $\underline{K(X)}$ ([H, Remark II.6.14.1 and Proposition II.6.15]), so we get an isomorphism of groups

$$\{\text{Cartier divisors on } X, +\}/\underset{\text{lin}}{\equiv} \;\simeq \{\text{Invertible sheaves on } X, \otimes\}/\text{isom.} = \mathrm{Pic}(X).$$

In some sense, Cartier divisors and invertible sheaves are more or less the same thing. However, we will try to use as often as possible the (additive) language of divisors instead of that of invertible sheaves; this allows for example for **Q**-divisors (which have no analogs in terms of sheaves).

We will write $H^i(X, D)$ instead of $H^i(X, \mathscr{O}_X(D))$ and, if $\mathscr{F}$ is a coherent sheaf on X, $\mathscr{F}(D)$ instead of $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(D)$.

Assume that X is moreover normal. One has

$$H^0(X, \mathscr{O}_X(D)) \simeq \{f \in K(X) \mid f = 0 \ \text{ or } \ \operatorname{div}(f) + D \geq 0\}. \tag{1.3}$$

Indeed, if (U_i, f_i) represents D, and f is a non-zero rational function on X such that $\operatorname{div}(f) + D$ is effective, ff_i is regular on U_i (because X is normal!), and $f|_{U_i} = (ff_i)\frac{1}{f_i}$ defines a section of $\mathscr{O}_X(D)$ over U_i. Conversely, any global section of $\mathscr{O}_X(D)$ is a rational function f on X such that, on each U_i, the product $f|_{U_i} f_i$ is regular. Hence $\operatorname{div}(f) + D$ effective.

Remark 1.9 Let D be a non-zero effective Cartier divisor on X. If we still denote by D the subscheme of X that it defines (see 1.4), we have an exact sequence of sheaves[3]

$$0 \to \mathscr{O}_X(-D) \to \mathscr{O}_X \to \mathscr{O}_D \to 0.$$

Remark 1.10 Going back to Definition 1.2 of Cartier divisors, one checks that the morphism

$$\begin{array}{rcl} H^0(X, \underline{K(X)}^\times / \mathscr{O}_X^\times) & \longrightarrow & H^1(X, \mathscr{O}_X^\times) \\ D & \longmapsto & [\mathscr{O}_X(D)] \end{array}$$

induced by (1.2) is the coboundary of the long exact sequence in cohomology induced the short exact sequence

$$0 \to \mathscr{O}_X^\times \to \underline{K(X)}^\times \to \underline{K(X)}^\times / \mathscr{O}_X^\times \to 0.$$

Principal divisors correspond to the image of $\underline{K(X)}^\times$ in $H^0(X, \underline{K(X)}^\times / \mathscr{O}_X^\times)$.

Example 1.11 An integral hypersurface Y in $\mathbf{P}^n_{\mathbf{k}}$ corresponds to a prime ideal of height 1 in $\mathbf{k}[x_0, \ldots, x_n]$, which is therefore (since the ring $\mathbf{k}[x_0, \ldots, x_n]$ is factorial) principal. Hence Y is defined by one (homogeneous) irreducible equation f of degree d (called the *degree of* Y). This defines a surjective morphism

$$\{\text{Cartier divisors on } \mathbf{P}^n_{\mathbf{k}}\} \to \mathbf{Z}.$$

Since f/x_0^d is a rational function on $\mathbf{P}^n_{\mathbf{k}}$ with divisor $Y - dH_0$ (where H_0 is the hyperplane defined by $x_0 = 0$), Y is linearly equivalent to dH_0. Conversely, the divisor of any rational function on $\mathbf{P}^n_{\mathbf{k}}$ has degree 0 (because it is the quotient of two homogeneous polynomials of the same degree), hence we obtain an isomorphism

$$\operatorname{Pic}(\mathbf{P}^n_{\mathbf{k}}) \simeq \mathbf{Z}.$$

We denote by $\mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(d)$ the invertible sheaf corresponding to an integer d (it is $\mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(D)$ for any divisor D of degree d). One checks that the space of global sections of $\mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(d)$ is 0 for $d < 0$ and isomorphic to the vector space of homogeneous polynomials of degree d in $\mathbf{k}[x_0, \ldots, x_n]$ for $d \geq 0$. More intrinsically, for any finite dimensional $\mathbf{k}$-vector space W, one has

$$H^0(\mathbf{P}(W), \mathscr{O}_{\mathbf{P}(W)}(d)) = \begin{cases} \mathsf{Sym}^d W^\vee & \text{if } d \geq 0, \\ 0 & \text{if } d < 0. \end{cases}$$

[3] Let i be the inclusion of D in X. Since this is an exact sequence of sheaves *on* X, the sheaf on the right should be $i_* \mathscr{O}_D$ (a sheaf on X with support on D). However, it is customary to drop i_*. Note that as far as cohomology calculations are concerned, this does not make any difference ([H, Lemma III.2.10]).

Exercise 1.12 Let X be a normal integral scheme. Prove

$$\operatorname{Pic}(X \times \mathbf{P}^n_{\mathbf{k}}) \simeq \operatorname{Pic}(X) \times \mathbf{Z}.$$

(*Hint:* proceed as in [H, Proposition II.6.6 and Example II.6.6.1]). In particular,

$$\operatorname{Pic}(\mathbf{P}^m_{\mathbf{k}} \times \mathbf{P}^n_{\mathbf{k}}) \simeq \mathbf{Z} \times \mathbf{Z}.$$

This can be seen directly as in Example 1.11 by proving first that any hypersurface in $\mathbf{P}^m_{\mathbf{k}} \times \mathbf{P}^n_{\mathbf{k}}$ is defined by a *bihomogeneous* polynomial in $((x_0, \dots, x_m), (y_0, \dots, y_n))$.

1.13. Pullback. Let $\pi\colon Y \to X$ be a morphism between integral schemes and let D be a Cartier divisor on X. The pullback $\pi^*\mathscr{O}_X(D)$ is an invertible subsheaf of $K(Y)$ hence defines a linear equivalence class of divisors on Y (improperly) denoted by π^*D. Only the linear equivalence class of π^*D is well-defined in general; however, when D is a divisor (U_i, f_i) whose support does not contain the image $\pi(Y)$, the collection $(\pi^{-1}(U_i), f_i \circ \pi)$ defines a divisor π^*D in that class. In particular, it makes sense to restrict a Cartier divisor to a subvariety not contained in its support, and to restrict a Cartier divisor *class* to any subvariety.

1.3 Intersection of curves and divisors

1.14. Curves A curve is a projective variety of dimension 1. On a smooth curve C, a (Cartier) divisor D is just a finite formal linear combination of closed points $\sum_{p\in C} n_p p$. We define its degree to be the integer $\sum n_p[k(p) : \mathbf{k}]$. If D is effective ($n_p \geq 0$ for all p), we can view it as a 0-dimensional subscheme of X with (affine) support the set of points p for which $n_p > 0$, where it is defined by the ideal $\mathfrak{m}_{X,p}^{n_p}$ (see 1.4). We have

$$h^0(D, \mathscr{O}_D) = \sum_p \dim_{\mathbf{k}}(\mathscr{O}_{X,p}/\mathfrak{m}_{X,p}^{n_p}) = \sum_p n_p \dim_{\mathbf{k}}(\mathscr{O}_{X,p}/\mathfrak{m}_{X,p}) = \deg(D). \tag{1.4}$$

This justifies the seemingly strange definition of the degree.

One proves (see [H, Corollary II.6.10]) that the degree of the divisor of a regular function is 0, hence the degree factors through

$$\operatorname{Pic}(C) \simeq \{\text{Cartier divisors on } C\}/\underset{\text{lin}}{\equiv} \;\to \mathbf{Z}.$$

Let X be a variety. It will be convenient to define a *curve on* X as a morphism $\rho\colon C \to X$, where C is a smooth (projective) curve. Given any, possibly singular, curve in X, one may consider its normalization as a "curve on X." For any Cartier divisor D on X, we set

$$(D \cdot C) := \deg(\rho^*D). \tag{1.5}$$

This definition extends to $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisors, but this intersection number is then only a rational number in general.

An important remark is that when D is effective and $\rho(C)$ is not contained in its support, this number is non-negative (this is because the class ρ^*D can be represented by an effective divisor on C; see 1.13). In general, it is 0 whenever $\rho(C)$ does not meet the support of D.

If $\pi\colon X \to Y$ is a morphism between varieties, a curve $\rho\colon C \to X$ can also be considered as a curve on Y via the composition $\pi\rho\colon C \to Y$. If D is a Cartier divisor on Y, we have the so-called *projection formula*

$$(\pi^*D \cdot C)_X = (D \cdot C)_Y. \tag{1.6}$$

1.15. Numerically equivalent divisors. Let X be a variety. We say that two **Q**-Cartier **Q**-divisors D and D' on X are *numerically equivalent* if

$$(D \cdot C) = (D' \cdot C)$$

for all curves $C \to X$. Linearly equivalent Cartier divisors are numerically equivalent. Numerical equivalence classes of Cartier divisors form a torsion-free abelian group for the addition of divisors, denoted by $\mathrm{NS}(X)$ (the *Néron–Severi group* of X); it is a quotient of the Picard group $\mathrm{Pic}(X)$.

One can also define the numerical equivalence class of a **Q**-Cartier **Q**-divisor in the **Q**-vector space

$$\mathrm{NS}(X)_{\mathbf{Q}} := \mathrm{NS}(X) \otimes \mathbf{Q}.$$

Let $\pi\colon Y \to X$ be a morphism between varieties. By the projection formula (1.6), pullback of divisors induces a **Q**-linear map

$$\psi^*\colon\ \mathrm{NS}(Y)_{\mathbf{Q}} \to \mathrm{NS}(X)_{\mathbf{Q}}.$$

Theorem 1.16 *If X is a* proper *variety, the group* $\mathrm{NS}(X)$ *is free abelian of finite rank, called the* Picard number *of X and denoted by $\rho(X)$.*

This is proved in [Kl, Proposition 3, p. 334]. Over the complex numbers, we will see in Section 2.3 that $N^1(X)_{\mathbf{Q}}$ is a subspace of (the finite-dimensional vector space) $H^2(X, \mathbf{Q})$.

We will prove that many important properties of Cartier divisors are numerical, in the sense that they only depend on their numerical equivalence class in the Néron–Severi group.

Example 1.17 (Curves) If X is a curve, any curve $\rho\colon C \to X$ factors through the normalization $\nu\colon \widehat{X} \to X$ and, for any Cartier divisor D on X, one has $\deg(\rho^*D) = \deg(\nu^*D)\deg(\rho)$. The numerical equivalence class of a divisor is therefore given by its degree on $\widehat{X}$, hence

$$\mathrm{NS}(X) \xrightarrow{\sim} \mathbf{Z}.$$

Example 1.18 (Blow up of a point) One deduces from Example 1.11 isomorphisms

$$\operatorname{Pic}(\mathbf{P}^n_{\mathbf{k}}) \simeq \operatorname{NS}(\mathbf{P}^n_{\mathbf{k}}) \simeq \mathbf{Z}[H].$$

Let O be a point of $\mathbf{P}^n_{\mathbf{k}}$ and let $\varepsilon\colon \widetilde{\mathbf{P}}^n_{\mathbf{k}} \to \mathbf{P}^n_{\mathbf{k}}$ be its blow up. If H_0 is a hyperplane in $\mathbf{P}^n_{\mathbf{k}}$ which does not contain O, it can be defined as

$$\widetilde{\mathbf{P}}^n_{\mathbf{k}} = \{(x,y) \in \mathbf{P}^n_{\mathbf{k}} \times H_0 \mid x \in \langle Oy \rangle\}$$

and ε is the first projection. The fiber $E := \varepsilon^{-1}(O) \subset \widetilde{\mathbf{P}}^n_{\mathbf{k}} \simeq H_0$ is called the *exceptional divisor* of the blow up and ε induces an isomorphism $\widetilde{\mathbf{P}}^n_{\mathbf{k}} \smallsetminus E \xrightarrow{\sim} \mathbf{P}^n_{\mathbf{k}} \smallsetminus \{O\}$. Assume $n \geq 2$. By [H, Proposition II.6.5], we have isomorphisms $\operatorname{Pic}(\widetilde{\mathbf{P}}^n_{\mathbf{k}} \smallsetminus E) \simeq \operatorname{Pic}(\mathbf{P}^n_k \smallsetminus \{O\}) \simeq \operatorname{Pic}(\mathbf{P}^n_k)$ and an exact sequence

$$\mathbf{Z} \xrightarrow{\alpha} \operatorname{Pic}(\widetilde{\mathbf{P}}^n_{\mathbf{k}}) \to \operatorname{Pic}(\widetilde{\mathbf{P}}^n_{\mathbf{k}} \smallsetminus E) \to 0,$$

where $\alpha(m) = [mE]$.

Let L be a line contained in E and let $H \subset \mathbf{P}^n_{\mathbf{k}}$ be a hyperplane. If H does not contain O, we have $(\varepsilon^* H \cdot L) = 0$. If it does contain O, one checks that $\varepsilon^* H$ can be written as $H' + E$, where H' meets E along a hyperplane in E. In particular, $(H' \cdot L) = 1$. This implies

$$0 = (\varepsilon^* H \cdot L) = ((H' + E) \cdot L) = (E \cdot L) + 1,$$

hence $(E \cdot L) = -1$. In particular, the map α is injective and we obtain

$$\operatorname{Pic}(\widetilde{\mathbf{P}}^n_{\mathbf{k}}) \simeq \operatorname{NS}(\widetilde{\mathbf{P}}^n_{\mathbf{k}}) \simeq \mathbf{Z}[\varepsilon^* H] \oplus \mathbf{Z}[E].$$

1.4 Line bundles

A *line bundle* on a scheme X is a scheme L with a morphism $\pi\colon L \to X$ which is locally (on the base) "trivial", i.e., isomorphic to $\mathbf{A}^1_U \to U$, in such a way that the changes of trivializations are linear, i.e., given by $(x,t) \mapsto (x, \varphi(x)t)$, for some $\varphi \in \mathscr{O}_X^\times(U)$. A *section* of $\pi\colon L \to X$ is a morphism $s\colon X \to L$ such that $\pi \circ s = \mathrm{Id}_X$. One checks that the sheaf of sections of $\pi\colon L \to X$ is an invertible sheaf on X. Conversely, to any invertible sheaf $\mathscr{L}$ on X, one can associate a line bundle on X: if $\mathscr{L}$ is trivial on an affine cover (U_i), just glue the $\mathbf{A}^1_{U_i}$ together, using the g_{ij} of (1.1). It is common to use the words "invertible sheaf" and "line bundle" interchangeably.

Assume that X is integral and normal. A non-zero section s of a line bundle $L \to X$ defines an effective Cartier divisor on X (by the equation $s = 0$ on each affine open subset of X over which L is trivial), which we denote by $\operatorname{div}(s)$. With the interpretation (1.3), if D is a Cartier divisor on X and L is the line bundle associated with $\mathscr{O}_X(D)$, we have

$$\operatorname{div}(s) = \operatorname{div}(f) + D.$$

In particular, if D is effective, the function $f = 1$ corresponds to a section of $\mathscr{O}_X(D)$ with divisor D. In general, any non-zero rational function f on X can be seen as a (regular, nowhere vanishing) section of the line bundle $\mathscr{O}_X(-\operatorname{div}(f))$.

Example 1.19 Let $\mathbf{k}$ be a field and let W be a $\mathbf{k}$-vector space. We construct a line bundle $L \to \mathbf{P}(W)$ whose fiber above a point x of $\mathbf{P}(W)$ is the line ℓ_x of W represented by x by setting

$$L = \{(x, v) \in \mathbf{P}(W) \times W \mid v \in \ell_x\}.$$

On the standard open set U_i (defined after choice of a basis for W), L is defined in $U_i \times W$ by the equations $v_j = v_i x_j$, for all $j \neq i$. The trivialization on U_i is given by $(x, v) \mapsto (x, v_i)$, so that $g_{ij}(x) = x_i/x_j$, for $x \in U_i \cap U_j$. One checks that this line bundle corresponds to $\mathscr{O}_{\mathbf{P}W}(-1)$ (see Example 1.11).

1.5 Linear systems and morphisms to projective spaces

Let X be a normal variety. Let $\mathscr{L}$ be an invertible sheaf on X and let $|\mathscr{L}|$ be the set of (effective) divisors of non-zero global sections of $\mathscr{L}$. It is called the *linear system* associated with $\mathscr{L}$. The quotient of two sections which have the same divisor is a regular function on X which does not vanish. *If X is projective*, this quotient is constant and the map $\operatorname{div}\colon \mathbf{P}(H^0(X, \mathscr{L})) \to |\mathscr{L}|$ is therefore bijective.

Let D be a Cartier divisor on X. We write $|D|$ instead of $|\mathscr{O}_X(D)|$; it is the set of effective divisors on X which are linearly equivalent to D.

1.20. Morphisms to a projective space. We now come to a very important point: the link between morphisms from X to a projective space and vector spaces of sections of invertible sheaves on the normal projective variety X.

Let W be a $\mathbf{k}$-vector space of finite dimension and let $\psi\colon X \to \mathbf{P}(W)$ be a regular map. Consider the invertible sheaf $\mathscr{L} = \psi^*\mathscr{O}_{\mathbf{P}(W)}(1)$ and the linear map

$$H^0(\psi)\colon W^\vee \simeq H^0\big(\mathbf{P}(W), \mathscr{O}_{\mathbf{P}(W)}(1)\big) \to H^0(X, \mathscr{L}).$$

A section of $\mathscr{O}_{\mathbf{P}W}(1)$ vanishes on a hyperplane; its image by $H^0(\psi)$ is zero if and only if $\psi(X)$ is contained in this hyperplane. In particular, $H^0(\psi)$ is injective if and only if $\psi(X)$ is not contained in any hyperplane.

If $\psi\colon X \dashrightarrow \mathbf{P}(W)$ is only a rational map, it is defined on a dense open subset U of X, and we get as above a linear map $W^\vee \to H^0(U, \mathscr{L})$. If X is locally factorial, the invertible sheaf $\mathscr{L}$ is defined on U but extends to X (write $\mathscr{L} = \mathscr{O}_U(D)$ and take the closure of D in X) and, since X is normal, the restriction $H^0(X, \mathscr{L}) \to H^0(U, \mathscr{L})$ is bijective, so we get again a map $W^\vee \to H^0(X, \mathscr{L})$.

Conversely, starting from an invertible sheaf $\mathscr{L}$ on X and a finite-dimensional vector space V of sections of $\mathscr{L}$, we define a rational map

$$\psi_V\colon X \dashrightarrow \mathbf{P}(V^\vee)$$

(also denoted by $\psi_{\mathscr{L}}$ when $V = H^0(X, \mathscr{L})$) by associating to a point x of X the hyperplane of sections of V that vanish at x. This map is not defined at points where all sections in V

vanish (they are called *base-points* of V). If we choose a basis $(s_0,\dots,s_N)$ for V, we have also

$$\psi_V(x)=\big(s_0(x),\dots,s_N(x)\big),$$

where it is understood that the $s_j(x)$ are computed via the same trivialization of $\mathscr{L}$ in a neighborhood of x; the corresponding point of $\mathbf{P}^N_{\mathbf{k}}$ is independent of the choice of this trivialization.

These two constructions are inverse of one another. In particular, regular maps from X to a projective space, whose image is not contained in any hyperplane, correspond to base-point-free linear systems on X.

Example 1.21 (The rational normal curve) We saw in Example 1.11 that the vector space $H^0(\mathbf{P}^1_{\mathbf{k}},\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(m))$ has dimension $m+1$. A basis is given by $(s^m,s^{m-1}t,\dots,t^m)$. The corresponding linear system is base-point-free and induces a curve

$$\begin{array}{ccc}\mathbf{P}^1_{\mathbf{k}} & \longrightarrow & \mathbf{P}^m_{\mathbf{k}}\\ (s,t) & \longmapsto & (s^m,s^{m-1}t,\dots,t^m)\end{array}$$

whose image (the *rational normal curve*) can be defined by the vanishing of all 2×2-minors of the matrix

$$\begin{pmatrix} x_0 & \cdots & x_{m-1}\\ x_1 & \cdots & x_m\end{pmatrix}.$$

Example 1.22 (The Veronese surface) We saw in Example 1.11 that the vector space $H^0(\mathbf{P}^2_{\mathbf{k}},\mathscr{O}_{\mathbf{P}^2_{\mathbf{k}}}(2))$ has dimension 6. The corresponding linear system is base-point-free and induces a morphism

$$\begin{array}{ccc}\mathbf{P}^2_{\mathbf{k}} & \longrightarrow & \mathbf{P}^5_{\mathbf{k}}\\ (s,t,u) & \longmapsto & (s^2,st,su,t^2,tu,u^2)\end{array}$$

whose image (the *Veronese surface*) can be defined by the vanishing of all 2×2-minors of the symmetric matrix

$$\begin{pmatrix} x_0 & x_1 & x_2\\ x_1 & x_3 & x_4\\ x_2 & x_4 & x_5\end{pmatrix}.$$

Example 1.23 (Cremona involution) The rational map

$$\begin{array}{cccc}\mathbf{P}^2_{\mathbf{k}} & \dashrightarrow & \mathbf{P}^2_{\mathbf{k}} & \\ (s,t,u) & \longmapsto & (\frac{1}{s},\frac{1}{t},\frac{1}{u}) & =(tu,su,st)\end{array}$$

is defined everywhere except at the 3 points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. It is associated with the subspace $\langle tu,su,st\rangle$ of $H^0(\mathbf{P}^2_{\mathbf{k}},\mathscr{O}_{\mathbf{P}^2_{\mathbf{k}}}(2))$ (which is the space of all conics passing through these 3 points).

1.6 Globally generated sheaves

Let X be a $\mathbf{k}$-scheme of finite type. A coherent sheaf $\mathscr{F}$ is *generated by its global sections at a point* $x \in X$ (or *globally generated at* x) if the images of the global sections of $\mathscr{F}$ (i.e., elements of $H^0(X, \mathscr{F})$) in the stalk $\mathscr{F}_x$ generate that stalk as a $\mathscr{O}_{X,x}$-module. The set of points at which $\mathscr{F}$ is globally generated is the complement of the support of the cokernel of the *evaluation* map

$$\mathsf{ev}\colon H^0(X, \mathscr{F}) \otimes_{\mathbf{k}} \mathscr{O}_X \to \mathscr{F}.$$

It is therefore open. The sheaf $\mathscr{F}$ is *generated by its global sections* (or *globally generated*) if it is generated by its global sections at each point $x \in X$. This is equivalent to the surjectivity of ev, and to the fact that $\mathscr{F}$ is the quotient of a free sheaf.

Since closed points are dense in X, it is enough to check global generation at every closed point x. This is equivalent, by Nakayama's lemma, to the surjectivity of the $k(x)$-linear map

$$\mathsf{ev}_x\colon H^0(X, \mathscr{F}) \otimes k(x) \to H^0(X, \mathscr{F} \otimes k(x))$$

We sometimes say that $\mathscr{F}$ is *generated by finitely many global sections* (at $x \in X$) if there are $s_0, \dots, s_N \in H^0(X, \mathscr{F})$ such that the corresponding evaluation maps, where $H^0(X, \mathscr{F})$ is replaced with the vector subspace generated by $s_0, \dots, s_N$, are surjective.

Any quasi-coherent sheaf on an affine sheaf $X = \mathrm{Spec}(A)$ is generated by its global sections (such a sheaf can be written as $\widetilde{M}$, where M is an A-module, and $H^0(X, \widetilde{M}) = M$).

Any quotient of a globally generated sheaf has the same property. Any tensor product of globally generated sheaves has the same property. The restriction of a globally generated sheaf to a subscheme has the same property.

1.24. Globally generated invertible sheaves. An invertible sheaf $\mathscr{L}$ on X is generated by its global sections if and only if for each closed point $x \in X$, there exists a global section $s \in H^0(X, \mathscr{L})$ that does not vanish at x (i.e., $s_x \notin \mathfrak{m}_{X,x}\mathscr{L}_x$, or $\mathsf{ev}_x(s) \neq 0$ in $H^0(X, \mathscr{L} \otimes k(x)) \simeq k(x)$).

Another way to phrase this, using the constructions of 1.20, is to say that the invertible sheaf $\mathscr{L}$ is generated by finitely many global sections if and only if there exists a *morphism* $\psi\colon X \to \mathbf{P}^N_{\mathbf{k}}$ such that $\psi^*\mathscr{O}_{\mathbf{P}^N_{\mathbf{k}}}(1) \simeq \mathscr{L}$.[4]

If D is a Cartier divisor on X, the invertible sheaf $\mathscr{O}_X(D)$ is generated by its global sections (for brevity, we will sometimes say that D is generated by its global sections, or globally generated) if for any $x \in X$, there is a Cartier divisor on X, linearly equivalent to D, whose support does not contain x (use (1.3)).

Example 1.25 We saw in Example 1.11 that any invertible sheaf on the projective space $\mathbf{P}^n_{\mathbf{k}}$ (with $n > 0$) is of the type $\mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(d)$ for some integer d. This sheaf is not generated by

[4] If $s \in H^0(X, \mathscr{L})$, the subset $X_s = \{x \in X \mid \mathsf{ev}_x(s) \neq 0\}$ is open. A family $(s_i)_{i\in I}$ of sections generate $\mathscr{L}$ if and only if $X = \bigcup_{i \in I} X_{s_i}$. If X is noetherian and $\mathscr{L}$ is globally generated, it is generated by finitely many global sections.

its global sections for $d \leq 0$ because any global section is constant. However, when $d > 0$, the vector space $H^0(\mathbf{P}^n_{\mathbf{k}}, \mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(d))$ is isomorphic to the space of homogeneous polynomials of degree d in the homogeneous coordinates $x_0, \dots, x_n$ on $\mathbf{P}^n_{\mathbf{k}}$. At each point of $\mathbf{P}^n_{\mathbf{k}}$, one of these coordinates, say x_i, does not vanish, hence the section x_i^d does not vanish either. It follows that $\mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(d)$ is generated by its global sections if and only if $d > 0$.

1.7 Ample divisors

The following definition, although technical, is extremely important.

Definition 1.26 *A Cartier divisor D on a scheme X of finite type over a field is* ample *if, for every coherent sheaf $\mathscr{F}$ on X, the sheaf $\mathscr{F}(mD)$ is generated by its global sections for all m large enough.*

Any sufficiently high multiple of an ample divisor is therefore globally generated, but an ample divisor may not be globally generated (it may have no non-zero global sections).

Proposition 1.27 *Let D be a Cartier divisor on a scheme of finite type over a field. The following conditions are equivalent:*

(i) *D is ample;*
(ii) *pD is ample for all $p > 0$;*
(iii) *pD is ample for some $p > 0$.*

Proof. Both implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are trivial. Assume that pD is ample. Let $\mathscr{F}$ be a coherent sheaf. For each $i \in \{0, \dots, p-1\}$, the sheaf $\mathscr{F}(iD)(mpD) = \mathscr{F}((i + mp)D)$ is generated by its global sections for $m \gg 0$. It follows that $\mathscr{F}(mD)$ is generated by its global sections for all $m \gg 0$, hence D is ample. □

This proposition allows us to say that a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor is ample if some (integral) positive multiple is ample (all further positive multiples are then ample by the proposition). The restriction of an ample $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor to a closed subscheme is ample. The sum of two ample $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisors is still ample. The sum of an ample $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor and a globally generated Cartier divisor is ample. Any $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor on an affine scheme of finite type over a field is ample.

Proposition 1.28 *Let A and E be $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisors on a scheme of finite type over a field. If A is ample, so is $A + tE$ for all t rational small enough.*

Proof. Upon multiplying by a large positive integer, we may assume that D and E are Cartier divisors.

Since A is ample, $mA \pm E$ is globally generated for all $m \gg 0$ and $(m+1)A \pm E$ is then ample. We write $A \pm tE = (1 - t(m+1))A + t((m+1)A \pm E)$. When $0 < t < \frac{1}{m+1}$, this divisor is therefore ample. □

Here is the fundamental result that justifies the definition of ampleness.

Theorem 1.29 (Serre) *The hyperplane divisor on $\mathbf{P}^n_{\mathbf{k}}$ is ample.*

More precisely, for any coherent sheaf $\mathscr{F}$ on $\mathbf{P}^n_{\mathbf{k}}$, the sheaf $\mathscr{F}(m)$[5] *is generated by* finitely many *global sections for all $m \gg 0$.*

Proof. The restriction of $\mathscr{F}$ to each standard affine open subset U_i is generated by finitely many sections $s_{ik} \in H^0(U_i, \mathscr{F})$. We want to show that each $s_{ik}x_i^m \in H^0(U_i, \mathscr{F}(m))$ extends for $m \gg 0$ to a section t_{ik} of $\mathscr{F}(m)$ on $\mathbf{P}^n_{\mathbf{k}}$.

Let $s \in H^0(U_i, \mathscr{F})$. It follows from [H, Lemma II.5.3.(b)]) that for each j, the section

$$x_i^p s|_{U_i \cap U_j} \in H^0(U_i \cap U_j, \mathscr{F}(p))$$

extends to a section $t_j \in H^0(U_j, \mathscr{F}(p))$ for $p \gg 0$ (in other words, t_j restricts to $x_i^p s$ on $U_i \cap U_j$). We then have

$$t_j|_{U_i \cap U_j \cap U_k} = t_k|_{U_i \cap U_j \cap U_k}$$

for all j and k hence, upon multiplying again by a power of x_i,

$$x_i^q t_j|_{U_j \cap U_k} = x_i^q t_k|_{U_j \cap U_k}.$$

for $q \gg 0$ ([H, Lemma II.5.3.(a)]). This means that the $x_i^q t_j$ glue to a section t of $\mathscr{F}(p+q)$ on $\mathbf{P}^n_{\mathbf{k}}$ which extends $x_i^{p+q}s$.

We thus obtain finitely many global sections t_{ik} of $\mathscr{F}(m)$ which generate $\mathscr{F}(m)$ on each U_i hence on $\mathbf{P}^n_{\mathbf{k}}$. □

An important consequence of Serre's theorem is that a projective scheme over $\mathbf{k}$ (defined as a closed subscheme of some $\mathbf{P}^n_{\mathbf{k}}$) carries an effective ample divisor. We also have more.

Corollary 1.30 *A Cartier divisor on a projective variety is linearly equivalent to the difference of two effective Cartier divisors.*

Proof. Let D be a Cartier divisor on the variety X and let A be an effective ample divisor on X. For $m \gg 0$, the invertible sheaf $\mathscr{O}_X(D + mA)$ is generated by its global sections. In particular, it has a non-zero section; let E be its (effective) divisor. We have $D \underset{\text{lin}}{\equiv} E - mA$, which proves the proposition. □

Corollary 1.31 (Serre) *Let X be a projective $\mathbf{k}$-scheme and let $\mathscr{F}$ be a coherent sheaf on X. For all integers q,*

a) *the $\mathbf{k}$-vector space $H^q(X, \mathscr{F})$ has finite dimension;*
b) *the $\mathbf{k}$-vector spaces $H^q(X, \mathscr{F}(m))$ all vanish for $m \gg 0$.*

[5] This is the traditional notation for the tensor product $\mathscr{F} \otimes \mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(m)$, which is the same also as $\mathscr{F}(mH)$.

PROOF. Assume $X \subset \mathbf{P}^n_{\mathbf{k}}$. Since any coherent sheaf on X can be considered as a coherent sheaf on $\mathbf{P}^n_{\mathbf{k}}$ (with the same cohomology), we may assume $X = \mathbf{P}^n_{\mathbf{k}}$. For $q > n$, we have $H^q(X, \mathscr{F}) = 0$ and we proceed by descending induction on q.

By Theorem 1.29, there exist integers r and p and an exact sequence

$$0 \longrightarrow \mathscr{G} \longrightarrow \mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(-p)^r \longrightarrow \mathscr{F} \longrightarrow 0$$

of coherent sheaves on $\mathbf{P}^n_{\mathbf{k}}$. The vector spaces $H^q(\mathbf{P}^n_{\mathbf{k}}, \mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(-p))$ can be computed by hand and are all finite-dimensional. The exact sequence

$$H^q(\mathbf{P}^n_{\mathbf{k}}, \mathscr{O}_X(-p))^r \longrightarrow H^q(\mathbf{P}^n_{\mathbf{k}}, \mathscr{F}) \longrightarrow H^{q+1}(\mathbf{P}^n_{\mathbf{k}}, \mathscr{G})$$

yields a).

Again, direct calculations show that $H^q(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(m-p))$ vanishes for all $m > p$ and all $q > 0$. The exact sequence

$$H^q(\mathbf{P}^n_{\mathbf{k}}, \mathscr{O}_X(m-p))^r \longrightarrow H^q(\mathbf{P}^n_{\mathbf{k}}, \mathscr{F}(m)) \longrightarrow H^{q+1}(\mathbf{P}^n_{\mathbf{k}}, \mathscr{G}(m))$$

yields b). □

1.32. A cohomological characterization of ample divisors. A further consequence of Serre's theorem is an important characterization of ample divisors by the vanishing of higher cohomology groups.

Theorem 1.33 *Let X be a projective* **k***-scheme and let D be a Cartier divisor on X. The following properties are equivalent:*

(i) *D est ample;*
(ii) *for each coherent sheaf $\mathscr{F}$ on X, we have $H^q(X, \mathscr{F}(mD)) = 0$ for all $m \gg 0$ and all $q > 0$;*
(iii) *for each coherent sheaf $\mathscr{F}$ on X, we have $H^1(X, \mathscr{F}(mD)) = 0$ for all $m \gg 0$.*

PROOF. Assume D ample. For $m \gg 0$, the divisor mD is globally generated, by a finite number of sections (see footnote 4). It defines a morphism $\psi\colon X \to \mathbf{P}^N_{\mathbf{k}}$ such that $\psi^* H \underset{\text{lin}}{\equiv} mD$. This morphism has finite fibers: if it contracts a curve $C \subset X$ to a point, one has $mD|_C \underset{\text{lin}}{\equiv} 0$, which contradicts the fact $D|_C$ is ample. Since X is projective, ψ is finite.[6]

Let $\mathscr{F}$ be a coherent sheaf on X. The sheaf $\psi_*\mathscr{F}$ is then coherent ([H, Corollary II.5.20]). Since ψ is finite, if $\mathscr{U}$ is a covering of $\mathbf{P}^N_{\mathbf{k}}$ by affine open subsets, $\psi^{-1}(\mathscr{U})$ is a covering of X by affine open subsets ([H, Exercise II.5.17.(b)]) and, by definition of $\psi_*\mathscr{F}$, the associated cochain complexes are isomorphic. This implies

$$H^q(X, \mathscr{F}) \simeq H^q(\mathbf{P}^N_{\mathbf{k}}, \psi_*\mathscr{F})$$

[6]The very important fact that a projective morphism with finite fibers is finite is deduced in [H] from the difficult Main Theorem of Zariski. In our case, it can also be proved in an elementary fashion (see [D2, th. 3.28]).

for all integers q. By Corollary 1.31.b) and the projection formula ([H, Exercise II.5.1.(d)]), we have, for all $q > 0$ and $s \gg 0$,

$$0 = H^q(\mathbf{P}^N_{\mathbf{k}}, (\psi_* \mathscr{F})(sH)) \simeq H^q(X, \mathscr{F}(s\psi^* H))) = H^q(X, \mathscr{F}(smD)).$$

Applying this to each of the sheaves $\mathscr{F}, \mathscr{F}(D), \ldots, \mathscr{F}((m-1)D)$, we see that (ii) holds.

Condition (ii) trivially implies (iii).

Assume that (iii) holds. Let $\mathscr{F}$ be a coherent sheaf on X, let x be a closed point of X, and let $\mathscr{G}$ be the kernel of the surjection

$$\mathscr{F} \to \mathscr{F} \otimes k(x)$$

of $\mathscr{O}_X$-modules. Since (iii) holds, there exists an integer m_0 such that

$$H^1(X, \mathscr{G}(mD)) = 0$$

for all $m \geq m_0$ (note that the integer m_0 may depend on $\mathscr{F}$ and x). Since the sequence

$$0 \to \mathscr{G}(mD) \to \mathscr{F}(mD) \to \mathscr{F}(mD) \otimes k(x) \to 0$$

is exact, the evaluation

$$H^0(X, \mathscr{F}(mD)) \to H^0(X, \mathscr{F}(mD) \otimes k(x))$$

is surjective. This means that its global sections generate $\mathscr{F}(mD)$ in a neighborhood $U_{\mathscr{F},m}$ of x. In particular, there exists an integer m_1 such that $m_1 D$ is globally generated on $U_{\mathscr{O}_X, m_1}$. For all $m \geq m_0$, the sheaf $\mathscr{F}(mD)$ is globally generated on

$$U_x = U_{\mathscr{O}_X, m_1} \cap U_{\mathscr{F}, m_0} \cap U_{\mathscr{F}, m_0+1} \cap \cdots \cap U_{\mathscr{F}, m_0+m_1-1}$$

since it can be written as

$$(\mathscr{F}((m_0 + s)D)) \otimes \mathscr{O}_X(r(m_1 D))$$

with $r \geq 0$ and $0 \leq s < m_1$. Cover X with a finite number of open subsets U_x and take the largest corresponding integer m_0. This shows that D is ample and finishes the proof of the theorem. □

Corollary 1.34 *Let X and Y be projective* **k***-schemes and let $\psi \colon X \to Y$ be a morphism with finite fibers. Let A be an ample* **Q***-Cartier* **Q***-divisor on Y. Then the* **Q***-Cartier* **Q***-divisor $\psi^* A$ is ample.*

Proof. We may assume that A is a Cartier divisor. Let $\mathscr{F}$ be a coherent sheaf on X. We use the same tools as in the proof of the theorem: the sheaf $\psi_* \mathscr{F}$ is coherent and since A is ample, $H^q(X, \mathscr{F}(m\psi^* A)) \simeq H^q(Y, (\psi_* \mathscr{F})(mA))$ vanishes for all $q > 0$ and $m \gg 0$. By Theorem 1.33, $\psi^* A$ est ample. □

Exercise 1.35 In the situation of the corollary, if ψ is *not* finite, show that $\psi^* A$ is *not* ample.

Exercise 1.36 Let X be a projective $\mathbf{k}$-scheme, let $\mathscr{F}$ be a coherent sheaf on X, and let $A_1, \dots, A_r$ be ample Cartier divisors on X. Show that for each $i > 0$, the set

$$\{(m_1, \dots, m_r) \in \mathbf{N}^r \mid H^i(X, \mathscr{F}(m_1 A_1 + \cdots + m_r A_r)) \neq 0\}$$

is finite.

1.8 Ample divisors on curves

We define the (arithmetic) genus of a (projective) curve X over a field $\mathbf{k}$ by

$$g(X) := \dim_{\mathbf{k}}(H^1(X, \mathscr{O}_X)) =: h^1(X, \mathscr{O}_X).$$

Example 1.37 The curve $\mathbf{P}^1_{\mathbf{k}}$ has genus 0. This can be obtained by a computation in Čech cohomology: cover X with the two standard affine subsets U_0 and U_1. The Čech complex

$$H^0(U_0, \mathscr{O}_{U_0}) \oplus H^0(U_1, \mathscr{O}_{U_1}) \to H^0(U_{01}, \mathscr{O}_{U_{01}})$$

is $\mathbf{k}[t] \oplus \mathbf{k}[t^{-1}] \to \mathbf{k}[t, t^{-1}]$, hence the result.

Exercise 1.38 Show that the genus of a plane curve of degree d is $(d-1)(d-2)/2$ (*Hint:* assume that $(0,0,1)$ is not on the curve, cover it with the affine subsets U_0 and U_1 and compute the Čech cohomology groups as above).

The following theorem is extremely useful.[7]

Theorem 1.39 (Riemann–Roch theorem) *Let X be a smooth curve. For any divisor D on X, we have*

$$\chi(X, D) = \deg(D) + \chi(X, \mathscr{O}_X) = \deg(D) + 1 - g(X).$$

PROOF. By Proposition 1.30, we can write $D \underset{\text{lin}}{\equiv} E - F$, where E and F are effective (Cartier) divisors on X. Considering them as (0-dimensional) subschemes of X, we have exact sequences (see Remark 1.9)

$$\begin{array}{ccccccc} 0 \to & \mathscr{O}_X(E-F) & \to & \mathscr{O}_X(E) & \to & \mathscr{O}_F & \to 0 \\ 0 \to & \mathscr{O}_X & \to & \mathscr{O}_X(E) & \to & \mathscr{O}_E & \to 0 \end{array}$$

[7] This should really be called the Hirzebruch–Riemann–Roch theorem (or a (very) particular case of it). The original Riemann–Roch theorem is our Theorem 1.39 with the dimension of $H^1(X, \mathscr{L})$ replaced with that of its Serre-dual $H^0(X, \omega_X \otimes \mathscr{L}^{-1})$.

(note that the sheaf $\mathscr{O}_F(E)$ is isomorphic to $\mathscr{O}_F$, because $\mathscr{O}_X(E)$ is isomorphic to $\mathscr{O}_X$ in a neighborhood of the (finite) support of F, and similarly, $\mathscr{O}_E(E) \simeq \mathscr{O}_E$). As remarked in (1.4), we have

$$\chi(F, \mathscr{O}_F) = h^0(F, \mathscr{O}_F) = \deg(F).$$

Similarly, $\chi(E, \mathscr{O}_E) = \deg(E)$. This implies

$$\begin{aligned}
\chi(X, D) &= \chi(X, E) - \chi(F, \mathscr{O}_F) \\
&= \chi(X, \mathscr{O}_X) + \chi(E, \mathscr{O}_E) - \deg(F) \\
&= \chi(X, \mathscr{O}_X) + \deg(E) - \deg(F) \\
&= \chi(X, \mathscr{O}_X) + \deg(D)
\end{aligned}$$

and the theorem is proved. □

We can now characterize ample divisors on smooth curves.

Corollary 1.40 *A* $\mathbf{Q}$*-divisor* D *on a smooth curve is ample if and only if* $\deg(D) > 0$.

Proof. We may assume that D is a divisor. Let p be a closed point of the smooth curve X. If D is ample, $mD - p$ is linearly equivalent to an effective divisor for some $m \gg 0$, in which case

$$0 \le \deg(mD - p) = m \deg(D) - \deg(p),$$

hence $\deg(D) > 0$.

Conversely, assume $\deg(D) > 0$. By Riemann–Roch, we have $H^0(X, mD) \neq 0$ for $m \gg 0$, so, upon replacing D by a positive multiple, we can assume that D is effective. As in the proof of the theorem, we then have an exact sequence

$$0 \to \mathscr{O}_X((m-1)D) \to \mathscr{O}_X(mD) \to \mathscr{O}_D \to 0,$$

from which we get a surjection

$$H^1(X, (m-1)D)) \to H^1(X, mD) \to 0.$$

Since these spaces are finite-dimensional, this will be a bijection for $m \gg 0$, in which case we get a surjection

$$H^0(X, mD) \to H^0(D, \mathscr{O}_D).$$

In particular, the evaluation map ev_x (see Section 1.6) for the sheaf $\mathscr{O}_X(mD)$ is surjective at every point x of the support of D. Since it is trivially surjective for x outside of this support (it has a section with divisor mD), the sheaf $\mathscr{O}_X(mD)$ is globally generated.

Its global sections therefore define a morphism $\psi\colon X \to \mathbf{P}^N_{\mathbf{k}}$ such that $\mathscr{O}_X(mD) = \psi^*\mathscr{O}_{\mathbf{P}^N_{\mathbf{k}}}(1)$. Since $\mathscr{O}_X(mD)$ is non trivial, ψ is not constant, hence finite because X is a curve. But then, $\mathscr{O}_X(mD) = \psi^*\mathscr{O}_{\mathbf{P}^N_{\mathbf{k}}}(1)$ is ample (Corollary 1.34) hence D is ample. □

Exercise 1.41 Let X be a curve and let p be a closed point. Show that $X \smallsetminus \{p\}$ is affine.

1.9 Nef divisors

Let X be a projective $\mathbf{k}$-scheme and let D be an ample $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor on X. For any non-constant curve $\rho\colon C \to X$, the $\mathbf{Q}$-divisor ρ^*D is ample (Corollary 1.34) hence its degree $(D\cdot C)$ is positive. The converse is not quite true, although producing a counter-example at this point is a bit tricky. On the other hand, it is easier to control the following important property.

Definition 1.42 *A $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor D on a projective $\mathbf{k}$-scheme X is* nef[8] *if $(D\cdot C) \geq 0$ for every curve C on X.*

Being nef is by definition a numerical property for $\mathbf{Q}$-divisors (it only depends on the numerical equivalence class in the Néron–Severi group). One can define nef classes in the finite-dimensional $\mathbf{R}$-vector space $\mathrm{NS}(X)_{\mathbf{R}}$ and they obviously form a closed convex cone

$$\mathrm{Nef}(X) \subset \mathrm{NS}(X)_{\mathbf{R}}. \tag{1.7}$$

An ample divisor is nef. The restriction of a nef $\mathbf{Q}$-divisor to a closed subscheme is again nef. More generally, the pullback by any morphism of a nef $\mathbf{Q}$-divisor is again nef. More precisely, if $\psi\colon X \to Y$ is a morphism between projective $\mathbf{k}$-schemes and D is a nef $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor on Y, the pullback ψ^*D is nef and, if ψ is not finite (so that it contracts a curve C which will then satisfy $(\psi^*D\cdot C) = 0$), its class is on the boundary of the cone $\mathrm{Nef}(X)$.

Example 1.43 If X is a curve, one has an isomorphism $\deg\colon \mathrm{NS}(X)_{\mathbf{R}} \xrightarrow{\sim} \mathbf{R}$ given by the degree on the normalization (Example 1.17) and, tautologically,

$$\mathrm{Nef}(X) = \deg^{-1}(\mathbf{R}^{\geq 0}).$$

Example 1.44 One checks that in $\mathrm{NS}(\mathbf{P}^n_{\mathbf{k}})_{\mathbf{R}} \simeq \mathbf{R}[H]$, one has

$$\mathrm{Nef}(\mathbf{P}^n_{\mathbf{k}}) = \mathbf{R}^{\geq 0}[H].$$

Let $\varepsilon\colon \widetilde{\mathbf{P}}^n_{\mathbf{k}} \to \mathbf{P}^n_{\mathbf{k}}$ be the blow up of a point O, with exceptional divisor E, so that $\mathrm{NS}(\widetilde{\mathbf{P}}^n_{\mathbf{k}})_{\mathbf{R}} \simeq \mathbf{R}[\varepsilon^*H] \oplus \mathbf{R}[E]$ (Example 1.18). The plane closed convex cone $\mathrm{Nef}(\widetilde{\mathbf{P}}^n_{\mathbf{k}})$ is bounded by two half-lines. Note that ε^*H is nef, as the pullback of a nef divisor, but for any $t > 0$, one has (if L is a line in E) $((\varepsilon^*H + tE)\cdot L) = -t$, hence $\varepsilon^*H + tE$ is not nef and $\mathbf{R}^{\geq 0}[\varepsilon^*H]$ is one of these half-lines.

Alternatively, one could have argued that since ε is not finite, the class ε^*H is on the boundary of $\mathrm{Nef}(\widetilde{\mathbf{P}}^n_{\mathbf{k}})$. To find the other boundary half-line, recall that there is another morphism $\widetilde{\mathbf{P}}^n_{\mathbf{k}} \to \mathbf{P}^{n-1}_{\mathbf{k}}$. One checks that the class of the inverse image of the hyperplane in $\mathbf{P}^{n-1}_{\mathbf{k}}$ is $[\varepsilon^*H - E]$ (its intersection with the strict transform of a line passing through O is 0); it generates the other boundary half-line.

[8] This acronym comes from "numerically effective," or "numerically eventually free" (according to [R, D.1.3]).

Exercise 1.45 Let $\varepsilon\colon X \to \mathbf{P}^n_{\mathbf{k}}$ be the blow up of two distinct points, with exceptional divisors E_1 and E_2. If H is a hyperplane on $\mathbf{P}^n_{\mathbf{k}}$, prove

$$\begin{aligned} \mathrm{NS}(X)_{\mathbf{R}} &\simeq \mathbf{R}[\varepsilon^*H] \oplus \mathbf{R}[E_1] \oplus \mathbf{R}[E_2], \\ \mathrm{Nef}(X) &\simeq \mathbf{R}^{\geq 0}[\varepsilon^*H] \oplus \mathbf{R}^{\geq 0}[\varepsilon^*H - E_1] \oplus \mathbf{R}^{\geq 0}[\varepsilon^*H - E_2]. \end{aligned}$$

1.10 Cones of divisors

One can also consider classes of ample Cartier divisors. They span a convex cone

$$\mathrm{Amp}(X) \subset \mathrm{Nef}(X) \subset \mathrm{NS}(X)_{\mathbf{R}}.$$

which is open by Proposition 1.28. However, we do not know whether ampleness is a numerical property: if the class of $\mathbf{Q}$-divisor D is in $\mathrm{Amp}(X)$, is D ample? This is an important but hard question. To answer it (positively), we will skip a whole chunk of the theory and accept without proof the following key result.

Theorem 1.46 *On a projective scheme, the sum of two* $\mathbf{Q}$*-Cartier* $\mathbf{Q}$*-divisors, one nef and one ample, is ample.*

Corollary 1.47 *On a projective scheme, ampleness is a numerical property and the ample cone is the interior of the nef cone.*

PROOF. Let X be a projective scheme and let D be a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor whose class is in the interior of the nef cone. We want to prove that D is ample. We may assume that D is a Cartier divisor. Let H be an ample divisor on X. Since $[D]$ is in the interior of the nef cone, $[D] - t[H]$ is still in the nef cone for some $t > 0$ (small enough). Then $D - tH$ is nef, and $D = (D - tH) + tH$ is ample by the theorem. □

We complete our collection of cones of divisors with the (convex) *effective cone* $\mathrm{Eff}(X) \subset \mathrm{NS}(X)_{\mathbf{R}}$ generated by classes of effective Cartier divisors. It contains the ample cone (why?). It may happen that $\mathrm{Eff}(X)$ is not closed and we let $\mathrm{Psef}(X)$, the *pseudo-effective cone*, be its closure. Finally, the *big cone*

$$\mathrm{Big}(X) = \mathrm{Eff}(X) + \mathrm{Amp}(X) \subset \mathrm{Eff}(X) \tag{1.8}$$

is the interior of the pseudo-effective cone. All in all, we have

$$\begin{array}{ccccc} \mathrm{Nef}(X) & \subset & \mathrm{Psef}(X) & \subset & \mathrm{NS}(X)_{\mathbf{R}} \\ & & \cup & & \\ \cup & & \mathrm{Eff}(X) & & \\ & & \cup & & \\ \mathrm{Amp}(X) & \subset & \mathrm{Big}(X). & & \end{array} \tag{1.9}$$

Example 1.48 With the notation of Example 1.44, we have

$$\begin{aligned}
\mathrm{NS}(\widetilde{\mathbf{P}}^n_{\mathbf{k}})_{\mathbf{R}} &\simeq \mathbf{R}[\varepsilon^*H] \oplus \mathbf{R}[E],\\
\mathrm{Big}(\widetilde{\mathbf{P}}^n_{\mathbf{k}}) &\simeq \mathbf{R}^{>0}[\varepsilon^*H - E] \oplus \mathbf{R}^{>0}[E],\\
\mathrm{Eff}(\widetilde{\mathbf{P}}^n_{\mathbf{k}}) = \mathrm{Psef}(\widetilde{\mathbf{P}}^n_{\mathbf{k}}) &\simeq \mathbf{R}^{\geq 0}[\varepsilon^*H - E] \oplus \mathbf{R}^{\geq 0}[E],\\
\mathrm{Amp}(\widetilde{\mathbf{P}}^n_{\mathbf{k}}) &\simeq \mathbf{R}^{>0}[\varepsilon^*H] \oplus \mathbf{R}^{>0}[\varepsilon^*H - E],\\
\mathrm{Nef}(\widetilde{\mathbf{P}}^n_{\mathbf{k}}) &\simeq \mathbf{R}^{\geq 0}[\varepsilon^*H] \oplus \mathbf{R}^{\geq 0}[\varepsilon^*H - E].
\end{aligned}$$

Only the effective cone needs to be described: its description follows from the fact that if $D \subset \widetilde{\mathbf{P}}^n_{\mathbf{k}}$ is an effective divisor other than E, with class $a[\varepsilon^*H-E]+b[E]$, its intersection with a line contained in E (but not in D) is $a-b$, which must therefore be non-negative, and its intersection with the strict transform of a line passing through O is b, which must also be non-negative, hence $a \geq b \geq 0$. The effective cone is therefore contained in $\mathbf{R}^{\geq 0}[\varepsilon^*H-E]\oplus\mathbf{R}^{\geq 0}[E]$, and the reverse inclusion is obvious.

1.11 The canonical divisor

Let X be a smooth variety. We define the *canonical sheaf* ω_X as the determinant of the sheaf of differentials Ω_X. It is an invertible sheaf on X. A canonical divisor K_X is any Cartier divisor on X which defines ω_X.

When X is only normal, with regular locus $j\colon U \hookrightarrow X$, we define ω_X as the (not necessarily invertible) sheaf $j_*\omega_U$. A canonical divisor K_X is then any Weil divisor on X which restricts to a canonical divisor on U.

If $Y \subset X$ is a normal Cartier divisor in X, one has the *adjunction formula*

$$K_Y = (K_X + Y)|_Y. \tag{1.10}$$

Example 1.49 The canonical sheaf on $\mathbf{P}^n_{\mathbf{k}}$ is $\omega_{\mathbf{P}^n_{\mathbf{k}}} = \mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(-n-1)$. If follows from the adjunction formula (1.10) that for a normal hypersurface X of degree d in $\mathbf{P}^n_{\mathbf{k}}$, one has $\omega_X = \mathscr{O}_X(-n-1+d)$.

When X is projective of dimension n and smooth (or only Cohen–Macaulay), the canonical sheaf ω_X is a dualizing sheaf: for any locally free coherent sheaf $\mathscr{F}$ on X, there are isomorphisms (Serre duality)

$$H^i(X,\mathscr{F}) \simeq H^{n-i}(X,\mathscr{F}^\vee \otimes \omega_X)^\vee.$$

Chapter 2

Riemann–Roch theorems

If A is an ample Cartier divisor on a projective scheme X, the vector space $H^0(X, mA)$ contains, for $m \gg 0$, "enough elements" to induce a finite morphism $\psi_{mA} \colon X \to \mathbf{P}^N_{\mathbf{k}}$. Riemann–Roch theorems deal with estimations of the dimensions $h^0(X, mA)$ of these spaces of sections. We will not prove much here, but only state the main results.

When X is a smooth curve, the Riemann–Roch theorem is (Theorem 1.39)

$$\chi(X, mD) := h^0(X, mD) - h^1(X, mD) = m \deg(D) + \chi(X, \mathscr{O}_X) = m \deg(D) + 1 - g(X).$$

This is a polynomial of degree 1 in m. This fact is actually very general and we will use it to define the intersection of n Cartier divisors on a projective scheme of dimension n over a field.

2.1 Intersecting two curves on a surface

On a surface, curves and hypersurfaces are the same thing. Formula (1.5) therefore defines the intersection number of two curves on a surface. We want to give another, more concrete, interpretation of this number.

Let X be a smooth projective surface defined over an algebraically closed field $\mathbf{k}$ and let C_1 and C_2 be two curves on X with no common component. We would like to define the intersection number of C_1 and C_2 as the number of intersection points "counted with multiplicities."

One way to do that is to define the intersection multiplicity of C_1 and C_2 at a point x of $C_1 \cap C_2$. If f_1 and f_2 be respective generators of the ideals of C_1 and C_2 at x, this is

$$m_x(C_1 \cap C_2) = \dim_{\mathbf{k}} \mathscr{O}_{X,x}/(f_1, f_2).$$

By the Nullstellensatz, the ideal (f_1, f_2) contains a power of the maximal ideal $\mathfrak{m}_{X,x}$, hence the number $m_x(C_1 \cap C_2)$ is finite. It is 1 if and only if f_1 and f_2 generate $\mathfrak{m}_{X,x}$, which means that they form a system of parameters at x, i.e., that C_1 and C_2 meet transversally at x.

We then set

$$(C_1 \cdot C_2) = \sum_{x \in C_1 \cap C_2} m_x(C_1 \cap C_2). \tag{2.1}$$

Another way to understand this definition is to consider the scheme-theoretic intersection $C_1 \cap C_2$. It is a scheme whose support is finite, and by definition, $\mathscr{O}_{C_1 \cap C_2, x} = \mathscr{O}_{X,x}/(f_1, f_2)$. Hence,

$$(C_1 \cdot C_2) = h^0(X, \mathscr{O}_{C_1 \cap C_2}). \tag{2.2}$$

There is still another way to interpret this number.

Theorem 2.1 *Under the hypotheses above, we have*

$$(C_1 \cdot C_2) = \chi(X, -C_1 - C_2) - \chi(X, -C_1) - \chi(X, -C_2) + \chi(X, \mathscr{O}_X). \tag{2.3}$$

Proof. Let s_1 be a section of $\mathscr{O}_X(C_1)$ with divisor C_1 and let s_2 be a section of $\mathscr{O}_X(C_2)$ with divisor C_2. One checks that we have an exact sequence

$$0 \to \mathscr{O}_X(-C_1 - C_2) \xrightarrow{(s_2, -s_1)} \mathscr{O}_X(-C_1) \oplus \mathscr{O}_X(-C_2) \xrightarrow{\binom{s_1}{s_2}} \mathscr{O}_X \to \mathscr{O}_{C_1 \cap C_2} \to 0.$$

(Use the fact that the local rings of X are factorial and that local equations of C_1 and C_2 have no common factor.) The theorem follows. □

A big advantage is that the right side of (2.3) now makes sense for any (even non-effective) Cartier divisors C_1 and C_2. This is the approach we will take in the next section to generalize this definition in all dimensions.

Finally, we check that the definition (2.1) agrees with our former definition (1.5).

Lemma 2.2 *For any smooth curve $C \subset X$ and any divisor D on X, we have*

$$(D \cdot C) = \deg(D|_C).$$

Proof. We have exact sequences

$$0 \to \mathscr{O}_X(-C) \to \mathscr{O}_X \to \mathscr{O}_C \to 0$$

and

$$0 \to \mathscr{O}_X(-C - D) \to \mathscr{O}_X(-D) \to \mathscr{O}_C(-D|_C) \to 0,$$

which give

$$(D \cdot C) = \chi(C, \mathscr{O}_C) - \chi(C, -D|_C) = \deg(D|_C)$$

by the Riemann-Roch theorem on C. □

Example 2.3 If C_1 and C_2 are curves in $\mathbf{P}^2_{\mathbf{k}}$ of respective degress d_1 and d_2, we have (this is Bézout's theorem)

$$(C_1 \cdot C_2) = d_1 d_2.$$

Indeed, since $\chi(\mathbf{P}^2_{\mathbf{k}}, \mathscr{O}_{\mathbf{P}^2_{\mathbf{k}}}(-d)) = \binom{d}{2}$ for $d \geq 0$, Theorem 2.1 gives

$$(C_1 \cdot C_2) = \binom{d_1 + d_2}{2} - \binom{d_1}{2} - \binom{d_2}{2} + 1 = d_1 d_2.$$

2.2 General intersection numbers

Although we will not present it here, the proof of the following theorem is not particularly hard (it proceeds by induction on n, the case $n = 1$ being the Riemann–Roch Theorem 1.39).

Theorem 2.4 *Let X be a projective $\mathbf{k}$-scheme of dimension n.*

a) *Let D be a Cartier divisor on X. The function $m \longmapsto \chi(X, mD)$ takes the same values on $\mathbf{Z}$ as a polynomial $P(T) \in \mathbf{Q}[T]$ of degree $\leq n$. We define (D^n) to be $n!$ times the coefficient of T^n in P.*

b) *More generally, if $D_1, \ldots, D_r$ are Cartier divisors on X, the function*

$$(m_1, \ldots, m_r) \longmapsto \chi(X, m_1 D_1 + \cdots + m_r D_r)$$

takes the same values on $\mathbf{Z}^r$ as a polynomial $P(T_1, \ldots, T_r)$ with rational coefficients of total degree $\leq n$. When $r \geq n$, we define the intersection number

$$(D_1 \cdot \ldots \cdot D_r)$$

to be the coefficient of $T_1 \cdots T_r$ in P (it is 0 when $r > n$).

c) *The map*

$$(D_1, \ldots, D_n) \longmapsto (D_1 \cdot \ldots \cdot D_n)$$

is $\mathbf{Z}$-multilinear, symmetric, and takes integral values.

The intersection number only depends on the linear equivalence classes of the divisors D_i, since it is defined from the invertible sheaves $\mathscr{O}_X(D_i)$ but in fact only on the numerical equivalence class of the D_i. This follows from the fact that for any numerically trivial divisor D and any coherent sheaf $\mathscr{F}$ on X, we have $\chi(X, \mathscr{F}(D)) = \chi(X, \mathscr{F})$ ([Kl, Section 2, Theorem 1]).

Example 2.5 If X is a subscheme of $\mathbf{P}^N_{\mathbf{k}}$ of dimension n and if $H|_X$ is a hyperplane section of X, the intersection number $((H|_X)^n)$ is the degree of X as defined in [H, Section I.7]. In particular, $(H^n) = 1$ on $\mathbf{P}^n_{\mathbf{k}}$.

Example 2.6 If $D_1, \ldots, D_n$ are effective and meet properly in a finite number of points, and if $\mathbf{k}$ is algebraically closed, the intersection number does have a geometric interpretation as the number of points in $D_1 \cap \cdots \cap D_n$, counted with multiplicities. This is the length of the 0-dimensional scheme-theoretic intersection $D_1 \cap \cdots \cap D_n$ (see [Ko1, Theorem VI.2.8]; compare with (2.1) and (2.2)).

By multilinearity, we may define intersection numbers of $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisors. For example, let X be the cone in $\mathbf{P}^3_{\mathbf{k}}$ with equation $x_0 x_1 = x_2^2$ (its vertex is $(0, 0, 0, 1)$) and let

L be the line defined by $x_0 = x_2 = 0$ (compare with Example 1.6). Then $2L$ is a hyperplane section ($x_0 = 0$) of X, hence $(2L)^2 = \deg(X) = 2$. So we have $(L^2) = 1/2$.

Intersection numbers are seldom computed directly from the definition. Here are two useful tools.

Proposition 2.7 *Let $\pi\colon Y \to X$ be a surjective morphism between projective varieties and let $D_1, \dots, D_r$ be Cartier divisors on X with $r \geq \dim(Y)$.*

a) **(Restriction formula)** *If D_r is effective,*

$$(D_1 \cdot \ldots \cdot D_r) = (D_1|_{D_r} \cdot \ldots \cdot D_{r-1}|_{D_r}).$$

b) **(Pullback formula)** *We have*[1]

$$(\pi^* D_1 \cdot \ldots \cdot \pi^* D_r) = \deg(\pi)(D_1 \cdot \ldots \cdot D_r).$$

Example 2.8 Let again $\varepsilon\colon \widetilde{\mathbf{P}}^n_{\mathbf{k}} \to \mathbf{P}^n_{\mathbf{k}}$ be the blow up of a point, with exceptional divisor E (Example 1.18). If L is a line contained in $E \simeq \mathbf{P}^{n-1}_{\mathbf{k}}$, we saw in that example that $(E \cdot L) = -1$. Since $\operatorname{Pic}(E) \simeq \mathbf{Z}$, this implies $\mathscr{O}_{\widetilde{\mathbf{P}}^n_{\mathbf{k}}}(E)|_E = \mathscr{O}_E(-1)$. The restriction formula then gives

$$(E^n) = ((E|_E)^{n-1}) = (-1)^{n-1}.$$

On the other hand, the divisor class $[\varepsilon^* H - E]$ is the pullback of a hyperplane class via the second projection $\widetilde{\mathbf{P}}^n_{\mathbf{k}} \to \mathbf{P}^{n-1}_{\mathbf{k}}$ (Example 1.44). The pullback formula therefore gives

$$((\varepsilon^* H - E)^n) = 0. \tag{2.4}$$

We may choose H such that $\varepsilon^* H$ does not meet E. The restriction formula then gives $(D_1 \cdot \ldots \cdot D_{n-2} \cdot E \cdot \varepsilon^* H) = 0$ for all divisors $D_1, \dots, D_{n-2}$. Expanding (2.4), we get

$$((\varepsilon^* H)^n) + (-1)^n (E^n) = 0$$

and again, by the projection formula, $(E^n) = (-1)^{n-1}((\varepsilon^* H)^n) = (-1)^{n-1}(H^n) = (-1)^{n-1}$.

Corollary 2.9 *Let D be a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor on a projective variety X of dimension n.*

If D is ample, $(D^n) > 0$; if D is nef, $(D^n) \geq 0$.

Proof. If D is ample, the sections of mD define, for $m \gg 0$, a finite morphism $\psi\colon X \to \mathbf{P}^N_{\mathbf{k}}$ such that $\psi^* H \underset{\text{lin}}{\equiv} mD$. Since the image $\psi(X)$ has dimension n, we have (Example 2.5) $((H|_{\psi(X)})^n) = \deg(\psi(X)) > 0$. The projection formula then yields

$$((mD)^n) = ((\psi^* H|_{\psi(X)})^n),$$

hence $(D^n) > 0$.

If D is only nef, choose an ample divisor A on X. For all $t \in \mathbf{Q}^{>0}$, the $\mathbf{Q}$-divisor $D + tA$ is ample (Theorem 1.46), and we get $(D^n) \geq 0$ by letting t go to 0 in $((D + tA)^n) > 0$. □

[1] The degree $\deg(\pi)$ is the degree of the field extension $\pi^*\colon K(X) \hookrightarrow K(Y)$ if this extension is finite, and 0 otherwise.

Exercise 2.10 Let $D_1, \dots, D_n$ be $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisors on a projective variety. Prove the following:

a) if $D_1, \dots, D_n$ are ample, $(D_1 \cdot \ldots \cdot D_n) > 0$;

b) if $D_1, \dots, D_n$ are nef, $(D_1 \cdot \ldots \cdot D_n) \geq 0$.

2.3 Intersection of divisors over the complex numbers

Let X be a smooth projective complex manifold of dimension n. There is a short exact sequence of analytic sheaves

$$0 \to \underline{\mathbf{Z}} \xrightarrow{\cdot 2i\pi} \mathscr{O}_{X,\mathrm{an}} \xrightarrow{\exp} \mathscr{O}^*_{X,\mathrm{an}} \to 0$$

which induces a morphism

$$c_1 \colon H^1(X, \mathscr{O}^*_{X,\mathrm{an}}) \to H^2(X, \mathbf{Z})$$

called the *first Chern class.* So we can in particular define the first Chern class of an algebraic line bundle on X. Given divisors $D_1, \dots, D_n$ on X, the intersection product $(D_1 \cdot \ldots \cdot D_n)$ defined in Theorem 2.4 is the cup product

$$c_1(\mathscr{O}_X(D_1)) \smile \cdots \smile c_1(\mathscr{O}_X(D_n)) \in H^{2n}(X, \mathbf{Z}) \simeq \mathbf{Z}.$$

In particular, the degree of a divisor D on a curve $C \subset X$ is

$$c_1(\nu^* \mathscr{O}_X(D)) \in H^2(\widetilde{C}, \mathbf{Z}) \simeq \mathbf{Z}.$$

where $\nu \colon \widetilde{C} \to C$ is the normalization of C.

Remark 2.11 A theorem of Serre says that the canonical map $H^1(X, \mathscr{O}^*_X) \to H^1(X, \mathscr{O}^*_{X,\mathrm{an}})$ is bijective. In other words, isomorphism classes of holomorphic and algebraic line bundles on X are the same.

2.4 Asymptotic numbers of sections

Let D be a Cartier divisor on a projective $\mathbf{k}$-scheme X of dimension n. By definition, we have

$$\chi(X, mD) = \sum_{i=0}^{n} (-1)^i h^i(X, mD) = m^n \frac{(D^n)}{n!} + O(m^{n-1}).$$

The following proposition (whose proof proceeds again by induction on n) gives more information on each term $h^i(X, mD)$.

Proposition 2.12 *Let D be a Cartier divisor on a projective* **k***-scheme X of dimension n.*

a) *We have $h^i(X, mD) = O(m^n)$ for all $i \geq 0$.*

b) *If D is nef, we have $h^i(X, mD) = O(m^{n-1})$ for all $i > 0$, hence*

$$h^0(X, mD) = m^n \frac{(D^n)}{n!} + O(m^{n-1}).$$

In particular, if D is ample, we have by Corollary 2.9

$$\lim_{m \to +\infty} \frac{h^0(X, mD)}{m^n} > 0.$$

We defined in (1.8) a big class as the sum of an effective and an ample class. Let us say that a **Q**-Cartier **Q**-divisor D is big if it can be written as the sum of an effective and an ample divisor. This is a numerical property since ampleness is.

Proposition 2.13 *A Cartier divisor D on a projective* **k***-scheme X is big if and only if*

$$\limsup_{m \to +\infty} \frac{h^0(X, mD)}{m^n} > 0. \tag{2.5}$$

In particular, if a **Q***-Cartier* **Q***-divisor D is nef, it is big if and only if $(D^n) > 0$.*

The limsup in (2.5) is actually a limit, but this is difficult to prove.

Proof. If D is big, we write it as $D = E + A$, with E effective and A ample. We may assume that E and A are divisors. Then $h^0(X, mD) \geq h^0(X, mA)$ for all m and (2.5) follows.

Assume conversely that (2.5) holds. Let A be an ample effective divisor on X. The exact sequence

$$0 \to \mathscr{O}_X(mD - A) \to \mathscr{O}_X(mD) \to \mathscr{O}_A(mD|_A) \to 0$$

induces an exact sequence

$$0 \to H^0(X, mD - A) \to H^0(X, mD) \to H^0(A, mD|_A).$$

Since A has dimension $n - 1$, Proposition 2.12.a) gives $h^0(A, mD|_A) = O(m^{n-1})$, hence (2.5) implies $H^0(X, mD - A) \neq 0$ for infinitely many $m > 0$. For those m, we can write $mD - A \underset{\text{lin}}{\equiv} E$, with E effective, hence D is big. □

Ample divisors are nef and big, but not conversely (see Example 1.48). Nef and big divisors share many of the properties of ample divisors: for example, Proposition 2.12 shows that the dimensions of the spaces of sections of their successive multiples grow in the same fashion. They are however much more tractable; for instance, the pullback of a nef and big divisor by a generically finite morphism is still nef and big.

2.5 Kodaira dimension and Iitaka fibrations

Let D be a Cartier divisor on a projective $\mathbf{k}$-scheme X. If D is ample, the sections of mD define, for $m \gg 0$, a finite morphism $\psi_{mD} \colon X \to \mathbf{P}^N_{\mathbf{k}}$ (one can even show that ψ_{mD} is a closed embedding for all $m \gg 0$).

One may wonder what happens for a general (non-ample) Cartier divisor. When D is only big, we may write $m_0 D = E + A$, with E effective and A ample divisors, for some positive integer m_0. Among the sections of mm_0D, one then finds the sections of mA times s_E^m, where $\operatorname{div}(s_E) = E$. In other words, the composition of the rational map

$$\psi_{mm_0D} \colon X \dashrightarrow \mathbf{P}^N_{\mathbf{k}}$$

with a suitable linear projection $\mathbf{P}^N_{\mathbf{k}} \dashrightarrow \mathbf{P}^M_{\mathbf{k}}$ is the morphism

$$\psi_{mA} \colon X \longrightarrow \mathbf{P}^M_{\mathbf{k}}.$$

In particular, $\max_{m>0} \dim(\psi_{mD}(X)) = n$.

We make the following important definition.

Definition 2.14 (Kodaira dimension) *Let X be a projective normal variety and let D be a Cartier divisor on X. We define the* Kodaira dimension *of D by*

$$\kappa(D) := \max_{m>0} \dim(\psi_{mD}(X)).$$

We make the convention

$$\kappa(D) = -\infty \quad \Longleftrightarrow \quad \forall m > 0 \quad H^0(X, mD) = 0$$

and we also have

$$\kappa(D) = 0 \quad \Longleftrightarrow \quad \max_{m>0} h^0(X, mD) = 1.$$

We just saw that big divisors have maximal Kodaira dimension $n := \dim(X)$. Conversely, if D has maximal Kodaira dimension n, some m_0D defines a rational map $\psi \colon X \dashrightarrow \mathbf{P}^N_{\mathbf{k}}$ with image $Y := \psi(X)$ of dimension n. Let $U \subset X$ be the largest smooth open subset on which ψ is defined. Since X is normal, we have $\operatorname{codim}_X(X \smallsetminus U) \geq 2$ and we can write $m_0 D|_U \underset{\text{lin}}{\equiv} \psi|_U^* H + E$, with E effective Cartier divisor on U. Since X is normal, we have for all $m > 0$

$$h^0(X, mm_0D) = h^0(U, mm_0D|_U) \geq h^0(U, m\psi|_U^* H) \geq h^0(Y, mH|_Y) = \frac{\deg(Y)}{n!} m^n + O(m^{n-1}),$$

hence D is big.

The Cartier divisors with maximal Kodaira dimension are therefore exactly the big divisors, so this property is numerical. This is not the case in general: although $\kappa(D)$

only depends on the linear equivalence classes of the divisor D, since it is defined from the invertible sheaf $\mathscr{O}_X(D)$, it is not in general invariant under numerical equivalence. If D is a divisor of degree 0 on a curve X (so that $D \underset{\text{num}}{\equiv} 0$), one has $\kappa(D) = -\infty$ if $[D]$ is not a torsion element of $\operatorname{Pic}(X)$, and $\kappa(D) = 0$ otherwise.

When X is a smooth projective variety, the Kodaira dimension $\kappa(X)$ of the canonical divisor K_X (see Section 1.11) is an important invariant of the variety (called the Kodaira dimension of X). We say that X is *of general type* is K_X is big, i.e., if $\kappa(X) = \dim(X)$.

Examples 2.15 1) The Kodaira dimension of $\mathbf{P}^n_{\mathbf{k}}$ is $-\infty$.

2) If X is a smooth hypersurface of degree d in $\mathbf{P}^n_{\mathbf{k}}$, its Kodaira dimension is (see Example 1.49)

$$\kappa(X) = \begin{cases} -\infty & \text{if } d \le n; \\ 0 & \text{if } d = n+1; \\ \dim(X) = n-1 & \text{if } d > n+1. \end{cases}$$

3) If X is a curve, its Kodaira dimension is

$$\kappa(X) = \begin{cases} -\infty & \text{if } g(X) = 0; \\ 0 & \text{if } g(X) = 1; \\ 1 & \text{if } g(X) \ge 2. \end{cases}$$

There is a general structure theorem. For a Cartier divisor D on an integral projective scheme X, we define the set

$$N(D) := \{m \ge 0 \mid H^0(X, mD) \neq 0\}.$$

It is a semi-group hence, if $N(D) \neq 0$ (i.e., if $\kappa(D) \neq -\infty$), all sufficiently large elements of $N(D)$ are multiples of a single largest positive integer which we denote by $e(D)$. When D is ample, and even only big, one has $e(D) = 1$.

Example 2.16 Let Y be a projective variety, let A be an ample divisor on Y, let E be an elliptic curve, and let B be an element of order $m \ge 2$ in the group $\operatorname{Pic}(E)$. On $X := Y \times E$, the divisor $D := \mathsf{pr}_1^* A + \mathsf{pr}_2^* B$ has Kodaira dimension $\dim(X) - 1$ and $e(D) = m$.

For each $m \in N(D)$, one can construct the rational map

$$\psi_{mD} \colon X \dashrightarrow \mathbf{P}(H^0(X, mD)^\vee).$$

Theorem 2.17 (Iitaka fibration) *Let X be a projective normal variety and let D be a Cartier divisor on X. For all $m \in N(D)$ sufficiently large, the induced maps*

$$X \dashrightarrow \psi_{mD}(X)$$

are all birationally equivalent to a fixed algebraic fibration

$$X_\infty \longrightarrow Y_\infty$$

of normal projective varieties, with $\dim(Y_\infty) = \kappa(D)$ *and such that the restriction of* D *to a very general fiber has Kodaira dimension 0.*

To understand this theorem, we need to know what an algebraic fibration is. Essentially, we require the fibers to be connected, so when D is big (i.e., $\dim(X_\infty) = \dim(Y_\infty)$), the theorem says in particular that for each m sufficiently large, ψ_{mD} is birational onto its image.

The formal definition is the following.

Definition 2.18 *An* (algebraic) fibration *is a projective morphism* $\pi\colon X \to Y$ *between varieties such that* $\pi_*\mathscr{O}_X \simeq \mathscr{O}_Y$.

The composition of two fibrations is a fibration. When Y is normal, any projective birational map $\pi\colon X \to Y$ is a fibration.[2] Conversely, any fibration $\pi\colon X \to Y$ with $\dim(X) = \dim(Y)$ is birational (this follows for example from Proposition 2.19 below).

Since the closure of the image of a morphism π is defined by the ideal sheaf kernel of the canonical map $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$, a fibration is surjective and, by Zariski's Main Theorem, its fibers are connected ([H, Corollary III.11.3]) and even geometrically connected ([G1, III, Corollaire (4.3.12)]). When Y is normal, the converse is true in characteristic 0.[3] More generally, any projective morphism $\rho\colon X \to Y$ between varieties factors as

$$\rho\colon X \xrightarrow{\pi} Y' \xrightarrow{u} Y$$

where π is a fibration and u is finite. This is the Stein factorization and ρ is a fibration if and only if u is an isomorphism.

Proposition 2.19 *A surjective projective morphism* $\rho\colon X \to Y$ *between normal varieties is a fibration if and only if the corresponding finitely generated field extension* $\mathbf{k}(Y) \subset \mathbf{k}(X)$ *is algebraically closed.*

This proposition allows us to extend the definition of a fibration to rational maps between normal varieties.

[2]For any affine open subset $U \subset Y$, the ring extension $H^0(U, \mathscr{O}_Y) \subset H^0(U, \pi_*\mathscr{O}_X)$ is finite because π is projective and the quotient fields are the same because $\mathbf{k}(Y) = \mathbf{k}(X)$. Since $H^0(U, \mathscr{O}_Y)$ is integrally closed in $\mathbf{k}(Y)$, these rings are the same.

[3]In general, one needs to require that the generic fiber of π be geometrically integral. In positive characteristic, u might very well be a bijection without being an isomorphism (even if Y is normal: think of the Frobenius morphism).

Proof. If the extension $\mathbf{k}(Y) \subset \mathbf{k}(X)$ is algebraically closed, we consider the Stein factorization $\rho\colon X \to Y' \xrightarrow{u} Y$. The extension $\mathbf{k}(Y) \subset \mathbf{k}(Y')$ is finite, hence algebraic, hence trivial since $\mathbf{k}(Y') \subset \mathbf{k}(X)$. The morphism u is then finite and birational, hence an isomorphism (because Y is normal), and ρ is a fibration.

Assume conversely that ρ is a fibration. Any element of $\mathbf{k}(Y)$ algebraic over $\mathbf{k}(X)$ generates a finite extension $\mathbf{k}(X) \subset \mathbf{K}$ contained in $\mathbf{k}(Y)$ which corresponds to a rational factorization $\rho\colon X \overset{\pi}{\dashrightarrow} Y' \overset{u}{\dashrightarrow} Y$, where $\mathbf{K} = \mathbf{k}(Y')$ and u is generically finite. Replacing X and Y' by suitable modifications, we may assume that π and u are morphisms. Since X is normal, ρ is still a fibration hence, for any affine open subset $U \subset Y$, the inclusions

$$H^0(U, \mathscr{O}_Y) \subset H^0(U, u_*\mathscr{O}_{Y'}) \subset H^0(U, \rho_*\mathscr{O}_X) = H^0(U, \mathscr{O}_Y)$$

are equalities, hence u has degree 1 and $\mathbf{K} = \mathbf{k}(X)$. This proves that the extension $\mathbf{k}(Y) \subset \mathbf{k}(X)$ is algebraically closed. □

Remark 2.20 Here are some other properties of a fibration $\pi\colon X \to Y$.

- If X is normal, so is Y.
- For any Cartier divisor D on Y, one has $H^0(X, \pi^*D) \simeq H^0(Y, D)$ and $\kappa(X, \pi^*D) \simeq \kappa(Y, D)$.
- The induced map $\pi^*\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$ is injective.

Chapter 3

Rational curves on varieties

Mori proved in 1979 a conjecture of Hartshorne characterizing projective spaces as the only smooth projective varieties with ample tangent bundle ([Mo1]). The techniques that Mori introduced to solve this conjecture have turned out to have more far reaching applications than Hartshorne's conjecture itself and we will explain them in this chapter.

3.1 Parametrizing curves

Let $\mathbf{k}$ be a field and let C be a smooth (projective) curve over $\mathbf{k}$. Given a quasi-projective $\mathbf{k}$-variety X, we want to "parametrize" all curves $C \to X$. If T is a $\mathbf{k}$-scheme, a *family of curves from C to X parametrized by T* is a morphism $\rho\colon C \times T \to X$: for each closed point $t \in T$, one has a curve $c \mapsto \rho_t(c) := \rho(c,t)$ (defined over the field $k(t)$).

We want to construct a $\mathbf{k}$-scheme $\operatorname{Mor}(C,X)$ and a "universal family of curves"

$$\mathsf{ev}\colon C \times \operatorname{Mor}(C,X) \to X, \tag{3.1}$$

called the called *evaluation map,* such that for any $\mathbf{k}$-scheme T, the correspondance between

- morphisms $\varphi\colon T \to \operatorname{Mor}(C,X)$ and
- families $\rho\colon C \times T \to X$ of curves parametrized by T

obtained by sending φ to

$$\rho(c,t) = \mathsf{ev}(c, \varphi(t))$$

is one-to-one. In other words, any family of curves parametrized by T is pulled back from the universal family (3.1) by a uniquely defined morphism $T \to \operatorname{Mor}(C,X)$.

Taking $T = \operatorname{Spec}(\mathbf{k})$, we see that $\mathbf{k}$-points of $\operatorname{Mor}(C,X)$ should be in one-to-one correspondence with curves $C \to X$.

Taking $T = \operatorname{Spec}(\mathbf{k}[\varepsilon]/(\varepsilon^2))$, we see that the Zariski tangent space to $\operatorname{Mor}(C,X)$ at a $\mathbf{k}$-point $[\rho]$ is isomorphic to the space of extensions of ρ to morphisms

$$\rho_\varepsilon\colon C \times \operatorname{Spec}\mathbf{k}[\varepsilon]/(\varepsilon^2) \to X$$

which should be thought of as first-order infinitesimal deformations of ρ.

Theorem 3.1 (Grothendieck, Mori) *Let* $\mathbf{k}$ *be a field, let* C *be a smooth* $\mathbf{k}$*-curve, let* X *be a smooth quasi-projective* $\mathbf{k}$*-variety, and let* $\rho\colon C \to X$ *be a curve.*

a) *There exist a* $\mathbf{k}$*-scheme* $\operatorname{Mor}(C,X)$*, locally of finite type, which parametrizes morphisms from* C *to* X*.*

b) *The Zariski tangent space to* $\operatorname{Mor}(C,X)$ *at* $[\rho]$ *is isomorphic to* $H^0(C,\rho^*T_X)$*.*

c) *Locally around* $[\rho]$*, the scheme* $\operatorname{Mor}(C,X)$ *is defined by* $h^1(C,\rho^*T_X)$ *equations in a smooth* $\mathbf{k}$*-variety of dimension* $h^0(C,\rho^*T_X)$*.*[1] *In particular, by Riemann–Roch,*[2]

$$\dim_{[\rho]} \operatorname{Mor}(C,X) \ge \chi(C,\rho^*T_X) = -(K_X\cdot C) + (1-g(C))\dim(X). \tag{3.2}$$

We will not reproduce Grothendieck's general construction, since it is very nicely explained in [G2], and we will only explain the much easier case $C = \mathbf{P}^1_{\mathbf{k}}$.

Any $\mathbf{k}$-morphism $\rho\colon \mathbf{P}^1_{\mathbf{k}} \to \mathbf{P}^N_{\mathbf{k}}$ can be written as

$$\rho(u,v) = (F_0(u,v),\dots,F_N(u,v)), \tag{3.3}$$

where $F_0,\dots,F_N$ are homogeneous polynomials in two variables, of the same degree d, with no non-constant common factor in $\mathbf{k}[U,V]$. Morphisms $\mathbf{P}^1_{\mathbf{k}} \to \mathbf{P}^N_{\mathbf{k}}$ of degree d are therefore parametrized by the (open) complement $\operatorname{Mor}_d(\mathbf{P}^1_{\mathbf{k}},\mathbf{P}^N_{\mathbf{k}})$ in $\mathbf{P}(\mathbf{k}[U,V]_d^{N+1})$ of the union, for all $e\in\{1,\dots,d\}$, of the (closed) images of the morphisms

$$\begin{array}{rcl}\mathbf{P}(\mathbf{k}[U,V]_e)\times \mathbf{P}(\mathbf{k}[U,V]_{d-e}^{N+1}) & \longrightarrow & \mathbf{P}(\mathbf{k}[U,V]_d^{N+1}) \\ (G,(G_0,\dots,G_N)) & \longmapsto & (GG_0,\dots,GG_N).\end{array}$$

The evaluation map is

$$\begin{array}{rccc}\mathsf{ev}\colon & \mathbf{P}^1_{\mathbf{k}}\times \operatorname{Mor}_d(\mathbf{P}^1_{\mathbf{k}},\mathbf{P}^N_{\mathbf{k}}) & \longrightarrow & \mathbf{P}^N_{\mathbf{k}} \\ & \big((u,v),\rho\big) & \longmapsto & \big(F_0(u,v),\dots,F_N(u,v)\big).\end{array}$$

Example 3.2 In the case $d=1$, we can write $F_i(u,v) = a_iu+b_iv$, with $(a_0,\dots,a_N,b_0,\dots,b_N)$ in $\mathbf{P}^{2N+1}_{\mathbf{k}}$. The condition that $F_0,\dots,F_N$ have no common zeroes is equivalent to

$$\operatorname{rank}\begin{pmatrix} a_0 & \cdots & a_N \\ b_0 & \cdots & b_N\end{pmatrix} = 2.$$

Its (closed) complement Z in $\mathbf{P}^{2N+1}_{\mathbf{k}}$ is defined by the vanishing $\begin{vmatrix} a_i & a_j \\ b_i & b_j\end{vmatrix} = 0$ of all 2×2-minors. The evaluation map is

$$\begin{array}{rccc}\mathsf{ev}\colon & \mathbf{P}^1_{\mathbf{k}}\times (\mathbf{P}^{2N+1}_{\mathbf{k}}\smallsetminus Z) & \longrightarrow & \mathbf{P}^N_{\mathbf{k}} \\ & \big((u,v),(a_0,\dots,a_N,b_0,\dots,b_N)\big) & \longmapsto & \big(a_0u+b_0v,\dots,a_Nu+b_Nv\big).\end{array}$$

[1] In particular, a sufficient (but not necessary!) condition for $\operatorname{Mor}(C,X)$ to be smooth at $[\rho]$ is $H^1(C,\rho^*T_X)=0$.

[2] We are using here a generalization of Theorem 1.39 to locally free sheaves of any rank.

Finally, morphisms from $\mathbf{P}^1_{\mathbf{k}}$ to $\mathbf{P}^N_{\mathbf{k}}$ are parametrized by the disjoint union

$$\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, \mathbf{P}^N_{\mathbf{k}}) = \bigsqcup_{d \geq 0} \mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, \mathbf{P}^N_{\mathbf{k}}) \tag{3.4}$$

of quasi-projective schemes.

When X is a (closed) subscheme of $\mathbf{P}^N_{\mathbf{k}}$ defined by homogeneous equations $G_1, \dots, G_m$, morphisms $\mathbf{P}^1_{\mathbf{k}} \to X$ of degree d are parametrized by the subscheme $\mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, X)$ of $\mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, \mathbf{P}^N_{\mathbf{k}})$ defined by the equations

$$G_j(F_0, \dots, F_N) = 0 \qquad \text{for all} \;\; j \in \{1, \dots, m\}.$$

Again, morphisms from $\mathbf{P}^1_{\mathbf{k}}$ to X are parametrized by the disjoint union

$$\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X) = \bigsqcup_{d \geq 0} \mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, X)$$

of quasi-projective schemes.[3]

We now make a very important remark. Assume that X can be defined by homogeneous equations $G_1, \dots, G_m$ with coefficients in a subring R of $\mathbf{k}$. If $\mathfrak{m}$ is a maximal ideal of R, one may consider the reduction $X_{\mathfrak{m}}$ of X modulo $\mathfrak{m}$: this is the subscheme of $\mathbf{P}^N_{R/\mathfrak{m}}$ defined by the reductions of the G_j modulo $\mathfrak{m}$.

Because the equations defining the complement of $\mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, \mathbf{P}^N_{\mathbf{k}})$ in $\mathbf{P}(\mathbf{k}[U,V]_d^{N+1})$ have coefficients in $\mathbf{Z}$ and are the same for all fields, the scheme $\mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, X)$ is defined over R and $\mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, X_{\mathfrak{m}})$ is its reduction modulo $\mathfrak{m}$. In fancy terms, one may express this as follows: if $\mathscr{X}$ is a scheme over $\mathrm{Spec}\, R$, the R-morphisms $\mathbf{P}^1_R \to \mathscr{X}$ are parametrized by the R-points of a locally noetherian scheme

$$\mathrm{Mor}(\mathbf{P}^1_R, \mathscr{X}) \to \mathrm{Spec}\, R$$

and the fiber of a closed point $\mathfrak{m}$ is the space $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, \mathscr{X}_{\mathfrak{m}})$.

3.2 Free rational curves and uniruled varieties

Let X be a smooth $\mathbf{k}$-variety of dimension n and let $\rho\colon \mathbf{P}^1_{\mathbf{k}} \to X$ be a non-constant morphism (this is called a *rational curve* on X). Any locally free coherent sheaf on $\mathbf{P}^1_{\mathbf{k}}$ is isomorphic to a direct sum of invertible sheaves, hence we can write

$$\rho^* T_X \simeq \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a_n), \tag{3.5}$$

[3]When X is only quasi-projective, embed it into some projective variety $\overline{X}$. There is an evaluation morphism

$$\mathsf{ev}\colon \mathbf{P}^1_{\mathbf{k}} \times \mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, \overline{X}) \longrightarrow \overline{X}$$

and $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ is the complement in $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, \overline{X})$ of the image by the (proper) second projection of the closed subscheme $\mathsf{ev}^{-1}(\overline{X} \smallsetminus X)$.

with $a_1 \geq \cdots \geq a_n$ and $a_1 \geq 2$. If $H^1(\mathbf{P}^1_{\mathbf{k}}, \rho^* T_X)$ vanishes (this happens exactly when $a_n \geq -1$), the scheme $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ is smooth at $[\rho]$ (Theorem 3.1). We investigate a stronger condition.

Definition 3.3 Let X be a $\mathbf{k}$-variety. A rational curve $\rho\colon \mathbf{P}^1_{\mathbf{k}} \to X$ is *free* if its image is contained in the smooth locus of X and $\rho^* T_X$ is generated by its global sections (this happens exactly when $a_n \geq 0$).

By Theorem 3.1, the scheme $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ is smooth at points corresponding to free rational curves. The K_X-degree of a free curve is $-\sum a_i < 0$.

The importance of free curves comes from the following result. We will say that a $\mathbf{k}$-variety X is covered by rational curves, or is *uniruled,* if there is a $\mathbf{k}$-variety M and a dominant morphism

$$\mathbf{P}^1_{\mathbf{k}} \times M \to X$$

which does not contract $\mathbf{P}^1_{\mathbf{k}} \times \{m\}$ for some (hence for all) geometric point $m \in X(\overline{\mathbf{k}})$.[4] By the universal property, we may assume that M is a component of $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$. Using again this universal property, one checks that given any field extension L of $\mathbf{k}$, the $\mathbf{k}$-variety X is uniruled if and only if the L-variety X_L is uniruled.

If X is uniruled and $\mathbf{k}$ is algebraically closed, there is a rational curve through every point of X. The converse holds if $\mathbf{k}$ is uncountable (use the fact that $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ has countably many components). It is therefore common to work on an algebraically closed uncountable extension of the base field.

Proposition 3.4 *Let X be a smooth quasi-projective* $\mathbf{k}$*-variety.*

a) *If the rational curve $\rho\colon \mathbf{P}^1_{\mathbf{k}} \to X$ is free, the evaluation map*

$$\mathsf{ev}\colon \mathbf{P}^1_{\mathbf{k}} \times \mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X) \to X$$

is smooth at all points of $\mathbf{P}^1_{\mathbf{k}} \times \{[\rho]\}$, hence X is uniruled.

b) *Conversely, if X is uniruled and* $\mathbf{k}$ *is algebraically closed* of characteristic 0, *there is a* free *rational curve on X through a general point of X.*

Proof. The tangent map to ev at $(t, [\rho])$ is the map

$$\begin{array}{rcl} T_{\mathbf{P}^1_{\mathbf{k}},t} \oplus H^0(\mathbf{P}^1_{\mathbf{k}}, \rho^* T_X) & \longrightarrow & T_{X,\rho(t)} \simeq (\rho^* T_X)_t \\ (u, \sigma) & \longmapsto & T_t\rho(u) + \sigma(t). \end{array}$$

If ρ is free, this map is surjective because the evaluation map

$$H^0(\mathbf{P}^1_{\mathbf{k}}, \rho^* T_X) \longrightarrow (\rho^* T_X)_t$$

[4] This is automatic if $\dim(M) = \dim(X) - 1$ and we can always reduce to that case, but the seemingly more general definition we gave is more flexible.

is. Since $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ is smooth at $[\rho]$, the morphism ev is smooth at $(t,[\rho])$ and proves a).

Conversely, if ev is dominant, it is smooth at a general point $(t,[\rho])$ because we are in characteristic 0. This implies that the map

$$H^0(\mathbf{P}^1_{\mathbf{k}}, \rho^*T_X) \to (\rho^*T_X)_t / \operatorname{Im}(T_t\rho) \tag{3.6}$$

is surjective. There is a commutative diagram

$$\begin{array}{ccc} H^0(\mathbf{P}^1_{\mathbf{k}}, \rho^*T_X) & \xrightarrow{a} & (\rho^*T_X)_t \\ \uparrow & & \uparrow{\scriptstyle T_t\rho} \\ H^0(\mathbf{P}^1_{\mathbf{k}}, T_{\mathbf{P}^1_{\mathbf{k}}}) & \xrightarrow{a'} & T_{\mathbf{P}^1_{\mathbf{k}},t}. \end{array}$$

Since a' is surjective, the image of a contains $\operatorname{Im}(T_t\rho)$. Since the map (3.6) is surjective, a is surjective. Hence ρ^*T_X is generated by global sections at one point. It is therefore generated by global sections and ρ is free. □

Corollary 3.5 *If X is a smooth projective uniruled variety over a field of characteristic 0, the* plurigenera $p_m(X) := h^0(X, \mathscr{O}_X(mK_X))$ *vanish for all $m > 0$, i.e., $\kappa(X) = -\infty$.*

The converse is conjectured to hold and has been proved in dimensions ≤ 3 (for curves, it is obvious since $p_1(X)$ is the genus of X; for surfaces, we have the Castelnuovo criterion: $p_{12}(X) = 0$ if and only if X is birationally isomorphic to a ruled surface).

Proof. We may assume that the base field $\mathbf{k}$ is algebraically closed. By Proposition 3.4.b), there is a free rational curve $\rho\colon \mathbf{P}^1_{\mathbf{k}} \to X$ through a general point of X. Since ρ^*K_X has negative degree, any section of $\mathscr{O}_X(mK_X)$ must vanish on $\rho(\mathbf{P}^1_{\mathbf{k}})$, hence on a dense subset of X, hence on X. □

Example 3.6 Let $\rho\colon \mathbf{P}^1_{\mathbf{k}} \to X$ be a rational curve on a surface X and let $C \subset X$ be its image. If ρ is free, C "moves" on X by Proposition 3.4.a), hence $(C^2) \geq 0$. Conversely, if C is smooth and $(C^2) \geq 0$, the *normal exact sequence*

$$0 \to T_C \to T_X|_C \to \mathscr{O}_C(C) \to 0$$

implies that C is free.

Example 3.7 If $\varepsilon\colon \widetilde{\mathbf{P}}^2_{\mathbf{k}} \to \mathbf{P}^2_{\mathbf{k}}$ is the blow up of one point, the exceptional curve E is not free because $(E^2) = -1$. If $C \subset \widetilde{\mathbf{P}}^2_{\mathbf{k}}$ is any other smooth rational curve, we write $C \underset{\text{lin}}{\equiv} d\varepsilon^*H - mE$ (Example 1.18). From $(C\cdot E) \geq 0$ (because $C \neq E$) and $(C\cdot(\varepsilon^*H - E)) \geq 0$ (because $\varepsilon^*H - E$ is nef by Example 1.44), we get $d \geq m \geq 0$, hence $(C^2) = d^2 - m^2 \geq 0$ and C is free by Example 3.6.

Example 3.8 On the blow up X of $\mathbf{P}^2_{\mathbf{C}}$ at nine general points, there are countably many rational curves with self-intersection -1 ([H, Exercise V.4.15.(e)]) and none of these curves are free by Example 3.6. The inverse image on X of a general line of $\mathbf{P}^2_{\mathbf{C}}$ is free by Example 3.6 again, since its self-intersection is 1.

Exercise 3.9 Let X be a subscheme of $\mathbf{P}^N_{\mathbf{k}}$ defined by equations of degrees $d_1, \dots, d_s$ over an algebraically closed field. Assume $d_1 + \cdots + d_s < N$. Show that through any point of X, there is a line contained in X (we say that X *is covered by lines* and it is in particular uniruled; in characteristic 0, such a general line is free by Proposition 3.4.b), but in positive characteristic, it may happen that none of these lines are free).

3.3 Bend-and-break lemmas

In this section, we explain the techniques that Mori invented to prove the conjecture of Hartshorne mentioned in the introduction to this chapter. The main idea is that if a curve deforms on a projective variety while passing through a fixed point, it must at some point break up with at least one rational component, hence the name "bend-and-break."

We work over an *algebraically closed field* $\mathbf{k}$.

The first bend-and-break lemma (which we will not prove) can be found in [Mo1, Theorems 5 and 6]. It says that a curve deforming non-trivially while keeping a point fixed must break into several pieces, including a rational curve passing through the fixed point. We introduce one piece of notation: if c is a point of a curve C and x a point of a variety X, we let $\mathrm{Mor}(C, X; c \mapsto x)$ be the closed subscheme of $\mathrm{Mor}(C, X)$ that parametrizes morphisms $\rho\colon C \to X$ such that $\rho(c) = x$. In terms of the evaluation map (3.1), it is defined as

$$\mathrm{Mor}(C, X; c \mapsto x) := \mathsf{pr}_1\big((\{c\} \times \mathrm{Mor}(C, X)) \cap \mathsf{ev}^{-1}(x)\big). \tag{3.7}$$

Proposition 3.10 (Mori) *Let X be a projective variety defined over an algebraically closed field, let $\rho\colon C \to X$ be a smooth curve, and let c be a point on C. If $\dim_{[\rho]} \mathrm{Mor}(C, X; c \mapsto \rho(c)) \geq 1$, there exists a rational curve on X through $\rho(c)$.*

It follows from (3.7) that

$$\dim_{[\rho]} \mathrm{Mor}(C, X; c \mapsto \rho(c)) \geq \dim_{[\rho]} \mathrm{Mor}(C, X) - \dim(X).$$

According to (3.2), *when X is smooth along $\rho(C)$*, the hypothesis of the proposition is therefore fulfilled whenever

$$(-K_X \cdot C) \geq g(C) \dim(X) + 1. \tag{3.8}$$

Once we know there is a rational curve, it may under certain conditions be broken up into several components. This is the second bend-and-break lemma (which we will not prove either).

Proposition 3.11 (Mori) *Let X be a projective variety and let $\rho\colon \mathbf{P}^1_{\mathbf{k}} \to X$ be a rational curve, birational onto its image $C \subset X$. If*

$$\dim_{[\rho]}(\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; 0 \mapsto \rho(0), \infty \mapsto \rho(\infty))) \geq 2,$$

the curve C can be deformed to a connected union of at least two rational curves on X still passing through $\rho(0)$ and $\rho(\infty)$.

As above, *when X is smooth along $f(\mathbf{P}^1_{\mathbf{k}})$*, the hypothesis is fulfilled whenever

$$(-K_X \cdot \mathbf{P}^1_{\mathbf{k}}) \geq \dim(X) + 2. \tag{3.9}$$

3.4 Rational curves on Fano varieties

A Fano variety is a smooth projective variety X such that $-K_X$ is ample.

Example 3.12 The projective space $\mathbf{P}^n_{\mathbf{k}}$ is a Fano variety. More generally, any smooth complete intersection in $\mathbf{P}^n_{\mathbf{k}}$ defined by equations of degrees $d_1, \dots, d_s$ with $d_1 + \cdots + d_s \leq n$ is a Fano variety. A finite product of Fano varieties is a Fano variety.

We will apply the bend-and-break lemmas to show that any Fano variety X is covered by rational curves. Start from any curve $\rho\colon C \to X$; we want to show, using the estimate (3.8), that it deforms non-trivially while keeping a point fixed. Since $-K_X$ is ample, the intersection number $(-K_X \cdot C)$ is positive. But we need it to be greater than $g(C)\dim(X)$. Composing ρ with a cover $C' \to C$ of degree m multiplies $(-K_X \cdot C)$ by m, but also (roughly) multiplies $g(C)$ by m, *except in positive characteristic,* where the Frobenius morphism allows us to increase the degree of ρ without changing the genus of C. This gives in that case the required rational curve on X. Using the second bend-and-break lemma, we can bound the degree of this curve by a constant depending only on the dimension of X, and this is essential for the remaining step: reduction of the characteristic zero case to positive characteristic.

We explain this last step in a simple case. Assume for a moment that X and x are defined over $\mathbf{Z} \subset \mathbf{k}$; for almost all prime numbers p, the reduction of X modulo p is a Fano variety of the same dimension hence there is a rational curve (defined over the algebraic closure of the field $\mathbf{F}_p$) through x. This means that the scheme $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; 0 \to x)$, which is defined over $\mathbf{Z}$, has a geometric point modulo almost all primes p. Since we can moreover bound the degree of the curve by a constant independent of p, we are in fact in a quasi-projective subscheme of $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; 0 \to x)$, and this implies that it has a point over $\bar{\mathbf{Q}}$, hence over $\overline{\mathbf{k}}$. In general, X and x are only defined over some finitely generated ring but these ideas still work.

Theorem 3.13 (Mori) *Let X be a Fano variety of dimension $n > 0$ defined over an algebraically closed field. Through any point of X there is a rational curve of $(-K_X)$-degree at most $n + 1$.*

Even over **C**, there is no known proof of this theorem that uses only transcendental methods. A consequence of the theorem is that Fano varieties are uniruled (see Section 3.2). However, some Fano varieties contain no free rational curves ([Ko2]; by Proposition 3.4.b), this may only happen in positive characteristics).

PROOF. Assume that the field $\mathbf{k}$ has characteristic $p > 0$; choose a smooth curve $\rho\colon C \to X$ through a point x of X and a point c of C such that $f(c) = x$. Consider the ($\mathbf{k}$-linear) Frobenius morphism $C_1 \to C$;[5] it has degree p, but C_1 and C being isomorphic as abstract schemes have the same genus. Iterating the construction, we get a morphism $F_m\colon C_m \to C$ of degree p^m between curves of the same genus which we compose with ρ. Then

$$(-K_X \cdot C_m) - ng(C_m) = -p^m(K_X \cdot C) - ng(C)$$

is positive for m large enough. By Proposition 3.10 and (3.8), there exists a (birational) rational curve $\rho'\colon \mathbf{P}^1_{\mathbf{k}} \to X$, with say $\rho'(0) = x$. If

$$(-K_X \cdot \mathbf{P}^1_{\mathbf{k}}) \geq n + 2,$$

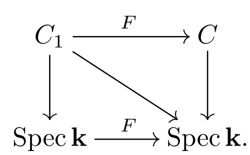
one can, by Proposition 3.11 and (3.8), break up the rational curve $\rho'(\mathbf{P}^1_{\mathbf{k}})$ into at least two (rational) pieces. Since $-K_X$ is ample, the component passing through x has smaller $(-K_X)$-degree, and we can repeat the process as long as $(-K_X \cdot \mathbf{P}^1_{\mathbf{k}}) \geq n + 2$, until we get to a rational curve of $(-K_X)$-degree no more than $n + 1$.

This proves the theorem in positive characteristic. Assume now that $\mathbf{k}$ has characteristic 0. Embed X in some projective space, where it is defined by a finite set of equations, and let R be the (finitely generated) subring of $\mathbf{k}$ generated by the coefficients of these equations and the coordinates of x.[6] There is a projective scheme $\mathscr{X} \to \operatorname{Spec}(R)$ with an R-point x_R, such that X is obtained from its generic fiber by base change from the quotient field $K(R)$ of R to $\mathbf{k}$. The geometric generic fiber is a Fano variety of dimension n, defined over the subfield $\overline{K(R)}$ of $\mathbf{k}$. There is a dense open subset U of $\operatorname{Spec}(R)$ over which $\mathscr{X}$ is smooth of dimension n ([G3, th. 12.2.4.(iii)]). Since ampleness is an open property ([G3, cor. 9.6.4]), we may even, upon shrinking U, assume that for each maximal ideal $\mathfrak{m}$ of R in U, the geometric fiber $X_{\mathfrak{m}}$ is a Fano variety of dimension n, defined over $\overline{R/\mathfrak{m}}$.

We will use the following two properties of the finitely generated domain R:[7]

- for each maximal ideal $\mathfrak{m}$ of R, the field $R/\mathfrak{m}$ is finite;

[5] If $F\colon \mathbf{k} \to \mathbf{k}$ is the Frobenius morphism, the $\mathbf{k}$-scheme C_1 fits into the Cartesian diagram



In other words, C_1 is the scheme C, but $\mathbf{k}$ acts on $\mathscr{O}_{C_1}$ via pth powers.

[6] This ring was $\mathbf{Z}$ in the brief description of the proof before the statement of the theorem.

[7] The first item is proved as follows. The field $R/\mathfrak{m}$ is a finitely generated $(\mathbf{Z}/\mathbf{Z} \cap \mathfrak{m})$-algebra, hence is finite over the quotient field of $\mathbf{Z}/\mathbf{Z} \cap \mathfrak{m}$ by the Nullstellensatz (which says that if k is a field and K a finitely

- maximal ideals (i.e., closed points) are dense in Spec(R).

As proved in Section 3.1, there is a quasi-projective scheme

$$\rho\colon \operatorname{Mor}_{\leq n+1}(\mathbf{P}^1_R, \mathscr{X}; 0 \mapsto x_R) \to \operatorname{Spec}(R)$$

which parametrizes rational curves of degree at most $n+1$ on $\mathscr{X}$ through x_R.

Let $\mathfrak{m}$ be a maximal ideal of R. Since the field $R/\mathfrak{m}$ is finite, hence of positive characteristic, what we just saw implies that the (geometric) fiber of ρ over a closed point of the dense open subset U of Spec(R) is nonempty; it follows that the image of ρ, which is a constructible[8] subset of Spec(R) by Chevalley's theorem ([H, Exercise II.3.19]), contains all closed points of U, therefore is dense by the second item, hence contains the generic point ([H, Exercise II.3.18.(b)]). This implies that the generic fiber is nonempty; it has therefore a geometric point, which corresponds to a rational curve on X through x, of degree at most $n+1$, defined over $\overline{K(R)}$, hence over $\mathbf{k}$.[9] □

3.5 Rational curves on varieties whose canonical divisor is not nef

We proved in Theorem 3.13 that when X is a smooth projective variety (defined over an algebraically closed field) such that $-K_X$ is ample (i.e., when X is a Fano variety), there is a rational curve through any point of X. The theorem we state in this section considerably weakens the hypothesis: assuming only that K_X has negative degree on *one* curve C, it says that there is a rational curve through any point of C.

Note that the proof of Theorem 3.13 goes through in positive characteristic under this weaker hypothesis and does prove the existence of a rational curve through any point of C. However, to pass to the characteristic 0 case, one needs to bound the degree of this rational curve with respect to some ample divisor by some "universal" constant so that we deal only

generated k-algebra which is a field, K is a finite extension of k; see [M, Theorem 5.2]). If $\mathbf{Z}\cap\mathfrak{m} = 0$, the field $R/\mathfrak{m}$ is a finite dimensional $\mathbf{Q}$-vector space with basis say $(e_1, \dots, e_m)$. If $x_1, \dots, x_r$ generate the $\mathbf{Z}$-algebra $R/\mathfrak{m}$, there exists an integer q such that qx_j belongs to $\mathbf{Z}e_1 \oplus \cdots \oplus \mathbf{Z}e_m$ for each j. This implies

$$\mathbf{Q}e_1 \oplus \cdots \oplus \mathbf{Q}e_m = R/\mathfrak{m} \subset \mathbf{Z}[1/q]e_1 \oplus \cdots \oplus \mathbf{Z}[1/q]e_m,$$

which is absurd; therefore, $\mathbf{Z}/\mathbf{Z}\cap\mathfrak{m}$ is finite and so is $R/\mathfrak{m}$.

For the second item, we need to show that the intersection of all maximal ideals of R is $\{0\}$. Let a be a non-zero element of R and let $\mathfrak{n}$ be a maximal ideal of the localization R_a. The field $R_a/\mathfrak{n}$ is finite by the first item hence its subring $R/R\cap\mathfrak{n}$ is a finite domain hence a field. Therefore $R\cap\mathfrak{n}$ is a maximal ideal of R which is in the open subset Spec(R_a) of Spec(R) (in other words, $a \notin \mathfrak{n}$).

[8]A constructible subset is a finite union of locally closed subsets.

[9]The "universal" bound on the degree of the rational curve is essential for the proof.

For those who know some elementary logic, the statement that there exists a rational curve of $(-K_X)$-degree at most some constant on a projective Fano variety X is a first-order statement, so Lefschetz principle tells us that if it is valid on all algebraically closed fields of positive characteristics, it is valid over all algebraically closed fields.

with a quasi-projective part of a morphism space. Apart from these technical difficulties, the ideas are essentially the same as in Theorem 3.13. This theorem is the main result of [MiM]. We will not prove it.

Theorem 3.14 (Miyaoka-Mori) *Let X be a projective variety defined over an algebraically closed field, let H be an ample divisor on X, and let $\rho\colon C \to X$ be a smooth curve such that X is smooth along $\rho(C)$ and $(K_X \cdot C) < 0$. Given any point x on $\rho(C)$, there exists a rational curve Γ on X through x with*

$$(H \cdot \Gamma) \leq 2 \dim(X) \, \frac{(H \cdot C)}{(-K_X \cdot C)}.$$

When X is smooth, the rational curve can be broken up, using Proposition 3.11 and (3.9), into several pieces (of lower H-degrees) keeping any two points fixed (one of which being on $\rho(C)$), until one gets a rational curve Γ which satisfies $(-K_X \cdot \Gamma) \leq \dim(X) + 1$ in addition to the bound on the H-degree.

It is nevertheless useful to have a more general statement allowing X to be singular. It implies for example that a normal projective variety X with ample ($\mathbf{Q}$-Cartier) anticanonical divisor is covered by rational curves of $(-K_X)$-degree at most $2\dim(X)$.

Finally, a simple corollary of this theorem is that *the canonical divisor of a smooth projective complex variety which contains no rational curves is nef.*

Our next result generalizes Theorem 3.13 and shows that varieties with nef but not numerically trivial anticanonical divisor are also covered by rational curves. This class of varieties is much larger than the class of Fano varieties.

Theorem 3.15 *If X is a smooth projective variety with $-K_X$ nef,*

- *either K_X is numerically trivial,*
- *or X is uniruled.*

Proof. We may assume that the base field is algebraically closed and uncountable. Let H be the restriction to X of a hyperplane in some embedding $X \subset \mathbf{P}^N_{\mathbf{k}}$. Assume $(K_X \cdot H^{n-1}) = 0$, where $n := \dim(X)$. For any curve $C \subset X$, there exist hypersurfaces $H_1, \dots, H_{n-1}$ in $\mathbf{P}^N_{\mathbf{k}}$, of respective degrees $d_1, \dots, d_{n-1}$, such that the scheme-theoretic intersection $Z := X \cap H_1 \cap \cdots \cap H_{n-1}$ has pure dimension 1 and contains C. Since $-K_X$ is nef, we have

$$0 \leq (-K_X \cdot C) \leq (-K_X \cdot Z) = d_1 \cdots d_{n-1} (-K_X \cdot H^{n-1}) = 0,$$

hence K_X is numerically trivial and we are in the first case.

Assume now $(K_X \cdot H^{n-1}) < 0$. Let x be a point of X and let C be the normalization of the intersection of $n-1$ general hyperplane sections through x. By Bertini's theorem, C is an irreducible curve and $(K_X \cdot C) = (K_X \cdot H^{n-1}) < 0$. By Theorem 3.14, there is a rational curve on X which passes through x and we are in the second case. □

An abelian variety has trivial canonical divisor and contains no rational curves.

Exercise 3.16 Let X be a smooth projective variety with $-K_X$ big. Show that X is covered by rational curves.

Exercise 3.17 Let X be a smooth projective variety, let $Y \subset X$ be a smooth hypersurface, and let $C \to X$ be a curve such that $(K_X \cdot C) = 0$ and $(Y \cdot C) < 0$. Prove that X contains a rational curve.

Chapter 4

The cone theorem

4.1 Cone of curves

We copy the definition of Weil divisors and define a (real) 1-cycle on a projective scheme X as a formal linear combination $\sum_C t_C C$, where the C are (embedded, possibly singular) integral projective curves in X and the t_C are real numbers.[1] Say that two such 1-cycles C and C' on X are numerically equivalent if

$$(D \cdot C) = (D \cdot C')$$

for all Cartier divisors D on X. The quotient of the vector space of 1-cycles by this equivalence relation is a real vector space $N_1(X)_{\mathbf{R}}$ which is canonically dual to $\mathrm{NS}(X)_{\mathbf{R}}$, hence finite-dimensional (Theorem 1.16).

In $N_1(X)_{\mathbf{R}}$, we define the convex cone $\mathrm{NE}(X)$ of classes of effective 1-cycles and its closure $\overline{\mathrm{NE}}(X)$. The nef cone $\mathrm{Nef}(X) \subset \mathrm{NS}(X)_{\mathbf{R}} = N_1(X)_{\mathbf{R}}^{\vee}$ (defined in (1.7)) is then simply the dual cone to $\mathrm{NE}(X)$ (or to $\overline{\mathrm{NE}}(X)$). Since the ample cone is the interior of the nef cone (Corollary 1.47), we obtain, from an elementary general property of dual cones, a useful characterization of ample classes.

Theorem 4.1 (Kleiman) *Let X be a projective variety. A Cartier divisor D on X is ample if and only if $D \cdot z > 0$ for all nonzero z in $\overline{\mathrm{NE}}(X)$.*

4.2 Mori's Cone Theorem

We fix the following notation: if D is a divisor on X and S a subset of $N_1(X)_{\mathbf{R}}$, we set

$$S_{D\geq 0} = \{z \in S \mid D \cdot z \geq 0\}$$

[1] To be consistent with our previous definition of "curve on X", we should perhaps say instead that C is a smooth projective curve with a morphism $C \to X$ which is birational onto its image.

and similarly for $S_{D\leq 0}$, $S_{D>0}$ and $S_{D<0}$.

Roughly speaking, Mori's cone theorem describes, for a smooth projective variety X, the part of the cone $\overline{\mathrm{NE}}(X)$ which has negative intersection with the canonical class, i.e., $\overline{\mathrm{NE}}(X)_{K_X<0}$.

An *extremal ray* of a closed convex cone $V \subset \mathbf{R}^m$ is a half-line $\mathbf{R}^{\geq 0}x \subset V$ such that

$$\forall v, v' \in V \qquad v + v' \in \mathbf{R}^{\geq 0}x \implies v, v' \in \mathbf{R}^{\geq 0}x.$$

If V contains no lines, it is the convex hull of its extremal rays. By Theorem 4.1, this is the case for the cone $\overline{\mathrm{NE}}(X)$ when X is projective

Theorem 4.2 (Mori's Cone Theorem) *Let X be a smooth projective variety. There exists a countable family $(\Gamma_i)_{i\in I}$ of rational curves on X such that*

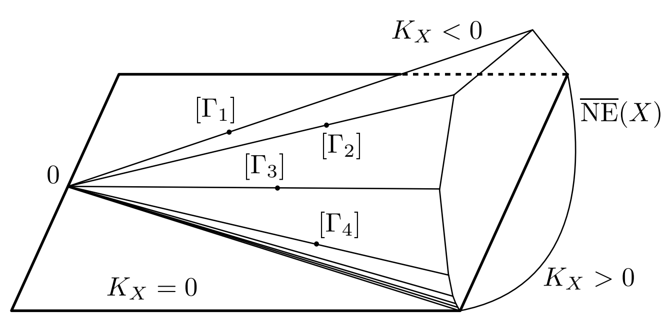
$$0 < (-K_X \cdot \Gamma_i) \leq \dim(X) + 1$$

and

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X\geq 0} + \sum_{i\in I} \mathbf{R}^{\geq 0}[\Gamma_i], \tag{4.1}$$

where the $\mathbf{R}^{\geq 0}[\Gamma_i]$ are all the extremal rays of $\overline{\mathrm{NE}}(X)$ that meet $N_1(X)_{K_X<0}$; these rays are locally discrete in that half-space.

An extremal ray that meets $N_1(X)_{K_X<0}$ is called *K_X-negative.*



The closed cone of curves

Sketch of proof. It follows from the description (3.4) of the scheme $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ that there are only countably many classes of rational curves on X. Pick a representative Γ_i for each such class z_i that satisfies $0 < -K_X \cdot z_i \leq \dim(X) + 1$.

First step: the rays $\mathbf{R}^{\geq 0}z_i$ are locally discrete in the half-space $N_1(X)_{K_X<0}$.

Let H be an ample divisor on X. If $z_i \in N_1(X)_{K_X+\varepsilon H<0}$, we have $(H \cdot \Gamma_i) < \frac{1}{\varepsilon}(-K_X \cdot \Gamma_i) \leq \frac{1}{\varepsilon}(\dim(X) + 1)$, and there are only finitely many such classes of curves on X (because the corresponding curves lie in a quasi-projective part of $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$).

Second step: $\overline{\mathrm{NE}}(X)$ *is equal to the closure of* $V = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_i \mathbf{R}^{\geq 0} z_i$. If not, there exists an $\mathbf{R}$-divisor M on X which is non-negative on $\overline{\mathrm{NE}}(X)$ (it is in particular nef), positive on $\overline{V} \smallsetminus \{0\}$ and which vanishes at some non-zero point z of $\overline{\mathrm{NE}}(X)$. This point cannot be in V, hence $K_X \cdot z < 0$.

Choose a norm on $N_1(X)_{\mathbf{R}}$ such that $\|[C]\| \geq 1$ for each irreducible curve C (this is possible since the set of classes of irreducible curves is discrete). We may assume, upon replacing M with a multiple, that $M \cdot v \geq 2\|v\|$ for all v in $\overline{V}$. We have

$$2\dim(X)(M \cdot z) = 0 < -K_X \cdot z.$$

Since the class $[M]$ is a limit of classes of ample $\mathbf{Q}$-divisors, and z is a limit of classes of effective rational 1-cycles, there exist an ample $\mathbf{Q}$-divisor H and an effective 1-cycle Z such that

$$2\dim(X)(H \cdot Z) < (-K_X \cdot Z) \qquad \text{and} \qquad H \cdot v \geq \|v\| \tag{4.2}$$

for all v in $\overline{V}$. We may further assume, by throwing away the other components, that each component C of Z satisfies $(-K_X \cdot C) > 0$.

Since the class of every rational curve Γ on X such that $(-K_X \cdot \Gamma) \leq \dim(X) + 1$ is in $\overline{V}$ (either it is in $\overline{\mathrm{NE}}(X)_{K_X \geq 0}$, or $(-K_X \cdot \Gamma) > 0$ and $[\Gamma]$ is one of the z_i), we have $(H \cdot \Gamma) \geq \|[\Gamma]\| \geq 1$ by (4.2) and the choice of the norm. Since X is smooth, the bend-and-break Theorem 3.14 implies

$$2\dim(X)\,\frac{(H \cdot C)}{(-K_X \cdot C)} \geq 1$$

for every component C of Z. This contradicts the first inequality in (4.2).

Third step: for any set J *of indices, the cone* $\overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_{j \in J} \mathbf{R}^{\geq 0} z_j$ *is closed.* We skip this relatively easy proof (a formal argument with no geometric content).

If we choose a set I of indices such that $(\mathbf{R}^{\geq 0} z_j)_{j \in I}$ is the set of all (distinct) extremal rays among all $\mathbf{R}^{\geq 0} z_i$, the proof shows that any extremal ray of $\overline{\mathrm{NE}}(X)_{K_X < 0}$ is spanned by a z_i, with $i \in I$. This finishes the proof of the cone theorem. ☐

Corollary 4.3 *Let* X *be a smooth projective variety and let* R *be a* K_X*-negative extremal ray. There exists a nef divisor* M_R *on* X *such that*

$$R = \{z \in \overline{\mathrm{NE}}(X) \mid M_R \cdot z = 0\}.$$

For any such divisor, $mM_R - K_X$ *is ample for all* $m \gg 0$.

Any such divisor M_R will be called a *supporting divisor* for R.

Proof. With the notation of the proof of the cone theorem, there exists a (unique) element i_0 of I such that $R = \mathbf{R}^{\geq 0} z_{i_0}$. By the third step of the proof, the cone

$$V = V_{I \smallsetminus \{i_0\}} = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_{i \in I,\, i \neq i_0} \mathbf{R}^{\geq 0} z_i$$

is closed and is strictly contained in $\overline{\mathrm{NE}}(X)$ since it does not contain R. This implies that there exists a linear form which is non-negative on $\overline{\mathrm{NE}}(X)$, positive on $V \smallsetminus \{0\}$ and which vanishes at some non-zero point of $\overline{\mathrm{NE}}(X)$, hence on R since $\overline{\mathrm{NE}}(X) = V + R$. The intersection of the interior of the dual cone V^* and the *rational* hyperplane $R^\perp$ is therefore nonempty, hence contains an integral point: there exists a divisor M_R on X which is positive on $V \smallsetminus \{0\}$ and vanishes on R. It is in particular nef and the first statement of the corollary is proved.

Choose a norm on $N_1(X)_{\mathbf{R}}$ and let a be the (positive) minimum of M_R on the set of elements of V with norm 1. If b is the maximum of K_X on the same compact, the divisor $mM_R - K_X$ is positive on $V \smallsetminus \{0\}$ for m rational greater than b/a, and positive on $R \smallsetminus \{0\}$ for $m \geq 0$, hence ample for $m > \max(b/a, 0)$ by Kleiman's criterion (Theorem 4.1). This finishes the proof of the corollary. □

Exercise 4.4 Let X be a smooth projective variety and let $A_1, \ldots, A_r$ be ample divisors on X. Show that $K_X + A_1 + \cdots + A_r$ is nef for all $r \geq \dim(X) + 1$.

Exercise 4.5 (A rationality result). Let X be a smooth projective variety whose canonical divisor is not nef and let M be a nef divisor on X. Set

$$r := \sup\{t \in \mathbf{R} \mid M + tK_X \text{ nef}\}.$$

a) Show that r is a (finite) non-negative real number.

b) Let $(\Gamma_i)_{i \in I}$ be the (nonempty and countable) set of rational curves on X that appears in the Cone Theorem 4.2. Show

$$r = \inf_{i \in I} \frac{(M \cdot \Gamma_i)}{(-K_X \cdot \Gamma_i)}.$$

c) Deduce that one can write $r = \frac{u}{v}$, with u and v relatively prime integers and $0 < v \leq \dim(X) + 1$, and that there exists a K_X-negative extremal ray R of $\overline{\mathrm{NE}}(X)$ such that

$$((M + rK_X) \cdot R) = 0.$$

4.3 Contractions of K_X-negative extremal rays

Let X be a smooth projective variety and let R be an extremal ray of $\overline{\mathrm{NE}}(X)$. A contraction of R is a fibration $c_R \colon X \twoheadrightarrow Y$ (see Definition 2.18) which contracts exactly those curves in X whose class is in R. A contraction can only exist when R is generated by the class of a curve; one can show that the contraction is then unique (up to isomorphisms of the bases).

The fact that K_X-negative extremal rays can be contracted is essential to the realization of Mori's minimal model program. This is only known in characteristic 0 (so say over $\mathbf{C}$) in all dimensions (and in any characteristic for surfaces) as a consequence of the following powerful theorem, whose proof is beyond the intended scope (and methods) of these notes.

Theorem 4.6 (Base-point-free theorem (Kawamata)) *Let X be a smooth complex projective variety and let D be a nef divisor on X such that $aD - K_X$ is nef and big for some $a \in \mathbf{Q}^{>0}$. Then mD is generated by its global sections for all $m \gg 0$.*

Corollary 4.7 *Let X be a smooth complex projective variety and let R be a K_X-negative extremal ray of $\overline{\mathrm{NE}}(X)$.*

a) *The contraction $c_R\colon X \twoheadrightarrow Y$ of R exists. It is given by the Stein factorization of the morphism defined by any sufficiently high multiple of any supporting divisor of R.*

b) *Let C be any curve on X with class in R. There is an exact sequence*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Pic}(Y) & \xrightarrow{c_R^*} & \mathrm{Pic}(X) & \longrightarrow & \mathbf{Z} \\ & & & & [D] & \longmapsto & (D\cdot C) \end{array}$$

and $\rho(Y) = \rho(X) - 1$.

c) *The restriction of $-K_X$ to any fiber of c_R is ample.*

Item b) implies

$$\mathrm{NS}(Y)_{\mathbf{R}} \xrightarrow{\sim} R^{\perp} \subset \mathrm{NS}(X)_{\mathbf{R}} \qquad \text{and} \qquad \mathrm{NS}(Y)_{\mathbf{R}} \xrightarrow{\sim} \mathrm{NS}(X)_{\mathbf{R}}/\langle R\rangle.$$

Proof of the corollary. Let M_R be a supporting divisor for R, as in Corollary 4.3. By the same corollary and Theorem 4.6, mM_R is generated by its global sections for $m \gg 0$. The contraction c_R is given by the Stein factorization of the induced morphism $X \to \mathbf{P}^N_{\mathbf{k}}$. This proves a). Note for later use that there exists a Cartier divisor D_m on Y such that $mM_R \underset{\text{lin}}{\equiv} c_R^* D_m$.

For b), we saw in Remark 2.20 that c_R^* is injective. Let now D be a divisor on X such that $(D\cdot C) = 0$. Proceeding as in the proof of Corollary 4.3, we see that the divisor $mM_R + D$ is nef for all $m \gg 0$ and vanishes only on R. It is therefore a supporting divisor for R hence some multiple $m'(mM_R + D)$ also defines its contraction. Since the contraction is unique, it is c_R and there exists a Cartier divisor $E_{m,m'}$ on Y such that $m'(mM_R + D) \underset{\text{lin}}{\equiv} c_R^* E_{m,m'}$. We obtain $D \underset{\text{lin}}{\equiv} c_R^*(E_{m,m'+1} - E_{m,m'} - D_m)$ and this finishes the proof of the corollary.

For c), let $F \subset X$ be a fiber of c_R and let $z \in \overline{\mathrm{NE}}(F)$ be non-zero. Since $z \in \overline{\mathrm{NE}}(X)$ and $mM_R - K_X$ is ample for m sufficiently large (Corollary 4.3), we have by Theorem 4.1 $(mM_R - K_X)\cdot z > 0$. Since $z \in \overline{\mathrm{NE}}(F)$, we have $M_R \cdot z = 0$, hence $(-K_X)\cdot z > 0$. By Theorem 4.1, this proves that $-K_X$ is ample on F. □

4.4 Various types of contractions

Let X be a smooth complex projective variety and let R be a K_X-negative extremal ray, with contraction $c_R\colon X \twoheadrightarrow Y$ (Corollary 4.7). The curves contracted by c_R are exactly those whose class is in R. Their union $\mathrm{locus}(R) \subset X$ is called the *locus* of R.

Since c_R is a fibration, either $\dim(Y) < \dim(X)$, or c_R is birational (Proposition 2.19). In the latter case, Zariski's Main Theorem says that $\text{locus}(R) = \pi^{-1}(\pi(\text{locus}(R)))$, the fibers of $\text{locus}(R) \to c_R(\text{locus}(R))$ are connected and everywhere positive-dimensional (in particular, $c_R(\text{locus}(R))$ has codimension at least 2 in Y), and c_R induces an isomorphism $X \smallsetminus \text{locus}(R) \xrightarrow{\sim} Y \smallsetminus c_R(\text{locus}(R))$.

There are 3 cases:

- $\text{locus}(R) = X$, so $\dim(c_R(X)) < \dim(X)$ and c_R is a *fiber contraction;*
- $\text{locus}(R)$ is a divisor in X and c_R is a *divisorial contraction;*
- $\text{locus}(R)$ has codimension at least 2 in X and c_R is a *small contraction.*

In the case of a divisorial contraction, the locus is always an irreducible divisor. In the case of a small contraction, the locus may be disconnected.

Proposition 4.8 *Let X be a smooth complex projective variety and let R be a K_X-negative extremal ray of $\overline{\text{NE}}(X)$ with contraction c_R. Then* locus(R) *is covered by* rational *curves contracted by c_R.*

Proof. Any point x in $\text{locus}(R)$ is on some irreducible curve C whose class is in R. Let M_R be a (nef) supporting divisor for R (as in Corollary 4.3), let H be an ample divisor on X, and let m be an integer such that

$$m > 2\dim(X)\frac{(H \cdot C)}{(-K_X \cdot C)}.$$

By Proposition 3.14, applied with the ample divisor $mM_R + H$, there exists a rational curve Γ through x such that

$$\begin{aligned} 0 &< ((mM_R + H)\cdot\Gamma) \\ &\leq 2\dim(X)\frac{((mM_R+H)\cdot C)}{(-K_X\cdot C)} \\ &= 2\dim(X)\frac{(H\cdot C)}{(-K_X\cdot C)} \\ &< m. \end{aligned}$$

It follows that the integer $(M_R \cdot \Gamma)$ must vanish, and $(H \cdot \Gamma) < m$: the class $[\Gamma]$ is in R hence Γ is contained in $\text{locus}(R)$. This proves the proposition. □

Exercise 4.9 Let $X \to \mathbf{P}^n_{\mathbf{k}}$ be the blow up of two distinct points. Determine the cone $\overline{\text{NE}}(X)$ and its extremal rays, and for each extremal ray, describe its contraction (see Exercise 1.45).

Exercise 4.10 Let X be a smooth projective variety of dimension n over an algebraically closed field of any characteristic. Let $R = \mathbf{R}^{\geq 0}z \subset \overline{\text{NE}}(X)$ be a K_X-negative extremal ray and let M_R be a supporting divisor for R (Corollary 4.3).

a) Let $C \subset X$ be an irreducible curve such that

$$(M_R \cdot C) < \frac{1}{2n}(-K_X \cdot C).$$

Prove that C is contained in locus(R).

b) If $(M_R^n) > 0$, prove that there exists an integral hypersurface $Y \subset X$ such that $Y \cdot z < 0$, hence locus$(R) \neq X$.

c) Show the converse: if locus$(R) \neq X$, then $(M_R^n) > 0$.

d) Explain why, in characteristic 0, c) and d) follow from the existence of a contraction of R (Corollary 4.7).

4.5 Fiber-type contractions

Let X be a smooth complex projective variety and let R be a K_X-negative extremal ray with contraction $c_R \colon X \twoheadrightarrow Y$ *of fiber type,* i.e., $\dim(Y) < \dim(X)$. It follows from Proposition 4.8.a) that X *is covered by rational curves* (contained in fibers of c_R). Moreover, a general fiber F of c_R is smooth and $-K_F = (-K_X)|_F$ is ample (Corollary 4.7.c)): F is a Fano variety as defined in Section 3.4.

The normal variety Y may be singular, but not too much. Recall that a variety is *locally factorial* if its local rings are unique factorization domains. This is equivalent to saying that all Weil divisors are Cartier divisors.

Proposition 4.11 *Let X be a smooth complex projective variety and let R be a K_X-negative extremal ray of $\overline{\mathrm{NE}}(X)$. If the contraction $c_R \colon X \twoheadrightarrow Y$ is of fiber type, Y is locally factorial.*

Proof. Let C be an irreducible curve whose class generates R (Theorem 4.2). Let D be a prime Weil divisor on Y. Let c_R^0 be the restriction of c_R to $c_R^{-1}(Y_{\mathrm{reg}})$ and let D_X be the closure in X of $(c_R^0)^*(D \cap Y_{\mathrm{reg}})$.

The Cartier divisor D_X is disjoint from a general fiber of c_R hence has intersection 0 with C. By Corollary 4.7.b), there exists a Cartier divisor D_Y on Y such that $D_X \underset{\mathrm{lin}}{\equiv} c_R^* D_Y$. Since $c_{R*}\mathscr{O}_X \simeq \mathscr{O}_Y$, by the projection formula, the Weil divisors D and D_Y are linearly equivalent on Y_{reg} hence on Y ([H, Proposition II.6.5.(b)]). This proves that D is a Cartier divisor and Y is locally factorial. □

Example 4.12 (A projective bundle is a fiber contraction) Let $\mathscr{E}$ be a locally free sheaf of rank $r \geq 2$ over a smooth projective variety Y and let $X = \mathbf{P}(\mathscr{E})$,[2] with projection $\pi \colon X \to Y$. If ξ is the class of the invertible sheaf $\mathscr{O}_X(1)$, we have

$$K_X = -r\xi + \pi^*(K_Y + \det(\mathscr{E})).$$

[2] We follow Grothendieck's notation: for a locally free sheaf $\mathscr{E}$, the projectivization $\mathbf{P}(\mathscr{E})$ is the space of *hyperplanes* in the fibers of $\mathscr{E}$.

If L is a line contained in a fiber of π, we have $(K_X \cdot L) = -r$. The class $[L]$ spans a K_X-negative ray whose contraction is π.

Example 4.13 (A fiber contraction which is not a projective bundle) Let C be a smooth curve of genus g, let d be a positive integer, and let $\operatorname{Pic}^d(C)$ be the Jacobian of C which parametrizes isomorphism classes of invertible sheaves of degree d on C.

Let C_d be the symmetric product of d copies of C; the Abel-Jacobi map $\pi_d\colon C_d \to \operatorname{Pic}^d(C)$ is a $\mathbf{P}^{d-g}$-bundle for $d \geq 2g-1$ hence is the contraction of a K_{C_d}-negative extremal ray by Example 4.12. In general, the fibers of π_d are still all projective spaces (of varying dimensions). If L_d is a line in a fiber, we have

$$(K_{C_d} \cdot L_d) = g - d - 1.$$

Indeed, the formula holds for $d \geq 2g-1$ by 4.12. Assume it holds for d; use a point of C to get an embedding $\iota\colon C_{d-1} \to C_d$. Then $(\iota^* C_{d-1} \cdot L_d) = 1$ and the adjunction formula yields

$$\begin{aligned}(K_{C_{d-1}} \cdot L_{d-1}) &= (\iota^*(K_{C_d} + C_{d-1}) \cdot L_{d-1}) \\ &= ((K_{C_d} + C_{d-1}) \cdot \iota_* L_{d-1}) \\ &= ((K_{C_d} + C_{d-1}) \cdot L_d), \\ &= (g-d-1)+1,\end{aligned}$$

which proves the formula by descending induction on d.

It follows that for $d \geq g$, the (surjective) map π_d is the contraction of the K_{C_d}-negative extremal ray $\mathbf{R}^{\geq 0}[L_d]$. It is a fiber contraction for $d > g$. For $d = g+1$, the generic fiber is $\mathbf{P}^1_{\mathbf{k}}$, but there are larger-dimensional fibers when $g \geq 3$, so the contraction is not a projective bundle.

4.6 Divisorial contractions

Let X be a smooth complex projective variety and let R be a K_X-negative extremal ray whose contraction $c_R\colon X \twoheadrightarrow Y$ is *divisorial.* It follows from Proposition 4.8.b) and its proof that the locus of R is an irreducible divisor E such that $E \cdot z < 0$ for all $z \in R \smallsetminus \{0\}$.

Again, Y may be singular (see Example 4.18), but not too much. We say that a scheme is *locally* $\mathbf{Q}$*-factorial* if any Weil divisor has a non-zero multiple which is a Cartier divisor. One can still intersect any Weil divisor D with a curve C on such a variety: choose a positive integer m such that mD is a Cartier divisor and set

$$(D \cdot C) = \frac{1}{m} \deg \mathscr{O}_C(mD).$$

This number is however only rational in general.

Proposition 4.14 *Let X be a smooth complex projective variety and let R be a K_X-negative extremal ray of $\overline{\mathrm{NE}}(X)$. If the contraction $c_R\colon X \twoheadrightarrow Y$ is divisorial, Y is locally $\mathbf{Q}$-factorial.*

PROOF. Let C be an irreducible curve whose class generates R (Theorem 4.2). Let D be a prime Weil divisor on Y. Let $c_R^0\colon c_R^{-1}(Y_{\mathrm{reg}}) \to Y_{\mathrm{reg}}$ be the morphism induced by c_R and let D_X be the closure in X of $c_R^{0*}(D \cap Y_{\mathrm{reg}})$.

Let E be the locus of R. Since $(E \cdot C) \neq 0$, there exist integers $a \neq 0$ and b such that $aD_X + bE$ has intersection 0 with C. By Corollary 4.7.b), there exists a Cartier divisor D_Y on Y such that $aD_X + bE \underset{\mathrm{lin}}{\equiv} c_R^* D_Y$.

Lemma 4.15 *Let X and Y be varieties, with Y normal, and let $\pi\colon X \to Y$ be a proper birational morphism. Let F an effective Cartier divisor on X whose support is contained in the exceptional locus of π. We have*

$$\pi_* \mathscr{O}_X(F) \simeq \mathscr{O}_Y.$$

PROOF. Since this is a statement which is local on Y, it is enough to prove $H^0(Y, \mathscr{O}_Y) \simeq H^0(Y, \pi_*\mathscr{O}_X(F))$ when Y is affine. By Zariski's Main Theorem, we have $H^0(Y, \mathscr{O}_Y) \simeq H^0(Y, \pi_*\mathscr{O}_X) \simeq H^0(X, \mathscr{O}_X)$, hence

$$H^0(Y, \mathscr{O}_Y) \simeq H^0(X, \mathscr{O}_X) \subset H^0(X, \mathscr{O}_X(F)) \subset H^0(X \smallsetminus E, \mathscr{O}_X(F))$$

and

$$H^0(X \smallsetminus E, \mathscr{O}_X(F)) \simeq H^0(X \smallsetminus E, \mathscr{O}_X) \simeq H^0(Y \smallsetminus \pi(E), \mathscr{O}_Y) \simeq H^0(Y, \mathscr{O}_Y),$$

the last isomorphism holding because Y is normal and $\pi(E)$ has codimension at least 2 in Y ([H, Exercise III.3.5]). All these spaces are therefore isomorphic, hence the lemma. □

Using the lemma, we get

$$\mathscr{O}_{Y_{\mathrm{reg}}}(D_Y) \simeq c^0_{R*}\mathscr{O}_{c_R^{-1}(Y_{\mathrm{reg}})}(aD_X + bE) \simeq \mathscr{O}_{Y_{\mathrm{reg}}}(aD) \otimes c^0_{R*}\mathscr{O}_{X^0}(bE) \simeq \mathscr{O}_{Y_{\mathrm{reg}}}(aD),$$

hence the Weil divisors aD and D_Y are linearly equivalent on Y. It follows that Y is locally $\mathbf{Q}$-factorial. □

Example 4.16 (A smooth blow up is a divisorial contraction) Let Y be a smooth projective variety, let Z be a smooth subvariety of Y of codimension c, and let $\pi\colon X \to Y$ be the blow up of Z, with exceptional divisor E. We have ([H, Exercise II.8.5.(b)])

$$K_X = \pi^* K_Y + (c-1)E.$$

Any fiber F of $E \to Z$ is isomorphic to $\mathbf{P}^{c-1}$ and $\mathscr{O}_F(E)$ is isomorphic to $\mathscr{O}_F(-1)$. If L is a line contained in F, we have $(K_X \cdot L) = -(c-1)$; the class $[L]$ therefore spans a K_X-negative ray whose contraction is π.

Example 4.17 (A divisorial contraction which is not a smooth blow up) We keep the notation of Example 4.13. The (surjective) map $\pi_g\colon C_g \twoheadrightarrow \mathrm{Pic}^g(C)$ is the contraction of the K_{C_g}-negative extremal ray $\mathbf{R}^{\geq 0}[L_g]$. Its locus is, by Riemann–Roch, the divisor

$$\{D \in C_g \mid h^0(C, K_C - D) > 0\}$$

and its image in $\mathrm{Pic}^g(C)$ has dimension $g-2$. The general fiber over this image is $\mathbf{P}^1_{\mathbf{k}}$, but there are bigger fibers when $g \geq 6$, because the curve C has a g^1_{g-2}, and the contraction is not a smooth blow up.

Example 4.18 (A divisorial contraction with singular image) Let Z be a smooth projective threefold and let C be an irreducible curve in Z whose only singularity is a node. The blow up Y of Z along C is normal and its only singularity is an ordinary double point q. This is checked by a local calculation: locally analytically, the ideal of C is generated by xy and z, where x, y, z form a system of parameters. The blow up is

$$\{((x,y,z),[u,v]) \in \mathbf{A}^3_{\mathbf{k}} \times \mathbf{P}^1_{\mathbf{k}} \mid xyv = zu\}.$$

It is smooth except at the point $q = ((0,0,0),[0,1])$. The exceptional divisor is the $\mathbf{P}^1_{\mathbf{k}}$-bundle over C with local equations $xy = z = 0$.

The blow up $\pi\colon X \to Y$ at q is smooth. It contains the proper transform E of the exceptional divisor of Y and an exceptional divisor Q, which is a smooth quadric. The intersection $E \cap Q$ is the union of two lines L_1 and L_2 belonging to the two different rulings of Q. Let $\tilde{E} \to E$ and $\tilde{C} \to C$ be the normalizations; each fiber of $\tilde{E} \to \tilde{C}$ is a smooth rational curve, except over the two preimages p_1 and p_2 of the node of C, where it is the union of two rational curves meeting transversally. Over p_i, one of these curves maps to L_i, the other one to the same rational curve L. *It follows that $L_1 + L$ and $L_2 + L$, hence also L_1 and L_2, are numerically equivalent on X*; they have the same class ℓ.

Any curve contracted by π is contained in Q hence its class is a multiple of ℓ. A local calculation shows that $\mathscr{O}_Q(K_X)$ is of type $(-1,-1)$, hence $K_X \cdot \ell = -1$. The ray $\mathbf{R}^{\geq 0}\ell$ is K_X-negative and its (divisorial) contraction is π (hence $\mathbf{R}^{\geq 0}\ell$ is extremal).[3]

Exercise 4.19 Let X be a smooth complex projective Fano variety with Picard number ≥ 2. Assume that X has an extremal ray whose contraction $X \to Y$ maps a hypersurface $E \subset X$ to a point. Show that X also has an extremal contraction whose fibers are all of dimension ≤ 1 (*Hint:* consider a ray R such that $(E \cdot R) > 0$.)

4.7 Small contractions and flips

Let X be a smooth complex projective variety and let R be a K_X-negative extremal ray whose contraction $c_R\colon X \twoheadrightarrow Y$ is *small*.

[3]This situation is very subtle: although the completion of the local ring $\mathscr{O}_{Y,q}$ is not factorial (it is isomorphic to $\mathbf{k}[[x,y,z,u]]/(xy-zu)$, and the equality $xy = zu$ is a decomposition in a product of irreducibles in two different ways) the fact that L_1 is numerically equivalent to L_2 implies that the ring $\mathscr{O}_{Y,q}$ is factorial (see [Mo2, (3.31)]).

The following proposition shows that Y is very singular: it is not even locally $\mathbf{Q}$-factorial, which means that one cannot intersect Weil divisors and curves on Y.

Proposition 4.20 *Let Y be a normal and locally $\mathbf{Q}$-factorial variety and let $\pi\colon X \to Y$ be a birational proper morphism. Every irreducible component of the exceptional locus of π has codimension 1 in X.*

Proof. Let E be the exceptional locus of π and let $x \in E$ and $y = \pi(x)$; identify the quotient fields $K(Y)$ and $K(X)$ by the isomorphism π^*, so that $\mathscr{O}_{Y,y} \subsetneq \mathscr{O}_{X,x} \subset K(X)$. Since $\mathscr{O}_{Y,y}/\mathfrak{m}_{Y,y} \simeq \mathscr{O}_{X,x}/\mathfrak{m}_{X,x} \simeq \mathbf{k}$ (because $\mathbf{k}$ is algebraically closed and x and y are closed points), there exists $t \in \mathfrak{m}_{X,x} \smallsetminus \mathscr{O}_{Y,y}$. Since $t \in K(Y)$, we may write its divisor on Y as the difference of two effective (Weil) divisors D' and D'' without common components.

Since Y is locally $\mathbf{Q}$-factorial, there exists a positive integer m such that mD' and mD'' are Cartier divisors, hence define elements u and v of $\mathscr{O}_{Y,y}$ such that $t^m = \frac{u}{v}$. Both are actually in $\mathfrak{m}_{Y,y}$: v because t^m is not in $\mathscr{O}_{Y,y}$ (otherwise, t would be since $\mathscr{O}_{Y,y}$ is integrally closed), and $u = t^m v$ because it is in $\mathfrak{m}_{X,x} \cap \mathscr{O}_{Y,y} = \mathfrak{m}_{Y,y}$. But $u = v = 0$ defines a subscheme Z of Y containing y of codimension 2 in some neighborhood of y (it is the intersection of the codimension 1 subschemes mD' and mD''), whereas $\pi^{-1}(Z)$ is defined by $t^m v = v = 0$ hence by the sole equation $v = 0$: it has codimension 1 in X, hence is contained in E. It follows that there is a codimension 1 component of E through every point of E, which proves the proposition. □

Since it is impossible to do anything useful with Y, Mori's idea is that there should exist instead another (mildly singular) projective variety X^+ with a small contraction $c^+\colon X^+ \to Y$ such that K_{X^+} has positive degree on curves contracted by c^+. The map c^+ (or sometimes the resulting rational map $(c^+)^{-1} \circ c\colon X \dashrightarrow X^+$) is called a *flip.*

Definition 4.21 *Let $c\colon X \twoheadrightarrow Y$ be a small contraction between normal projective varieties. Assume that K_X is $\mathbf{Q}$-Cartier and $-K_X$ is ample on all fibers of c. A* flip *of c is a small contraction $c^+\colon X^+ \to Y$ such that*

- *X^+ is a projective normal variety;*
- *K_{X^+} is $\mathbf{Q}$-Cartier and ample on all fibers of c^+.*

The *existence* of a flip of the small contraction of a negative extremal ray has only been shown recently ([BCHM]; see also [Dr, cor. 2.5]).

Proposition 4.22 *Let X be a locally $\mathbf{Q}$-factorial complex projective variety and let $c\colon X \twoheadrightarrow Y$ be a* small *contraction of a K_X-negative extremal ray R. If the flip $X^+ \twoheadrightarrow Y$ exists, the variety X^+ is locally $\mathbf{Q}$-factorial with Picard number $\rho(X)$.*

Proof. The composition $\varphi = c^{-1} \circ c^+\colon X^+ \dashrightarrow X$ is an isomorphism in codimension 1, hence induces an isomorphism between the Weil divisor class groups of the normal varieties

X and X^+ ([H, Proposition II.6.5.(b)]). Let D^+ be a Weil divisor on X^+ and let D be the corresponding Weil divisor on X. Let C be an irreducible curve whose class generates R, let r be a rational number such that $((D + rK_X) \cdot C) = 0$, and let m be an integer such that mD, mrK_X, and mrK_{X^+} are Cartier divisors (the fact that K_{X^+} is **Q**-Cartier is part of the definition of a flip!). By Corollary 4.7.b), there exists a Cartier divisor D_Y on Y such that $m(D + rK_X) \underset{\text{lin}}{\equiv} c^* D_Y$, and

$$mD^+ = \varphi^*(mD) \underset{\text{lin}}{\equiv} (c^+)^* D_Y - \varphi^*(mrK_X) \underset{\text{lin}}{\equiv} (c^+)^* D_Y - mrK_{X^+}$$

is a Cartier divisor. This proves that X^+ is locally **Q**-factorial. Moreover, φ^* induces an isomorphism between $N^1(X)_{\mathbf{R}}$ and $N^1(X^+)_{\mathbf{R}}$, hence the Picard numbers are the same. □

Contrary to the case of a divisorial contraction, the Picard number stays the same after a flip. So the second main problem is the *termination* of flips: can there exist an infinite chain of flips? It is conjectured that the answer is negative, but this is still unknown in general.

Exercise 4.23 Let V be a **k**-vector space of dimension n and let $r \in \{1, \dots, n-1\}$. Let $\mathsf{Gr}(r, V)$ be the *Grassmannian* that parametrizes vector subspaces of V of codimension r and set

$$X := \{(W, [u]) \in \mathsf{Gr}(r, V) \times \mathbf{P}(\mathrm{End}(V)) \mid u(W) = 0\}.$$

a) Show that X is smooth irreducible of dimension $r(2n-r)-1$, that $\mathrm{Pic}(X) \simeq \mathbf{Z}^2$, and that the projection $\mathsf{pr}_1 \colon X \to \mathsf{Gr}(r, V)$ is a K_X-negative extremal contraction.

b) Show that

$$Y := \mathsf{pr}_2(X) = \{[u] \in \mathbf{P}(\mathrm{End}(V)) \mid \mathrm{rank}(u) \leq r\}$$

is irreducible of dimension $r(2n - r) - 1$. It can be proved that Y is normal. If $r \geq 2$, show that Y is not locally **Q**-factorial and that $\mathrm{Pic}(Y) \simeq \mathbf{Z}[\mathscr{O}_Y(1)]$. What happens when $r = 1$?

4.8 The minimal model program

Given a projective variety X defined over an algebraically closed field **k**, one may try to find another projective variety birationally isomorphic to X and which is "as simple as possible." More formally, we define, on the set $\mathscr{C}_X$ of all (isomorphism classes of) projective varieties birationally isomorphic to X, a relation as follows: if X_0 and X_1 are in $\mathscr{C}_X$, we write $X_0 \succeq X_1$ if there is a birational *morphism* $X_0 \to X_1$. This defines an ordering on $\mathscr{C}_X$ and we look for *minimal elements* in $\mathscr{C}_X$ or even, if we are optimisitic, for the *smallest element* of $\mathscr{C}_X$.

When X is a smooth surface, it has a smooth minimal model obtained by contracting all exceptional curves on X. If X is not uniruled, this minimal model has nef canonical divisor and is the smallest element in $\mathscr{C}_X$. When X is uniruled, this minimal model is not unique, and is either a ruled surface or $\mathbf{P}^2_{\mathbf{k}}$.

The next proposition (which we will not prove) shows that smooth projective varieties with nef canonical bundles are minimal in the above sense. They are called *minimal models.*

Proposition 4.24 *Let X and Y be smooth projective varieties and let $\pi\colon X \to Y$ be a birational morphism which is not an isomorphism. There exists a rational curve C on X contracted by π such that $(K_X \cdot C) < 0$.*

In particular, if K_X is nef, X is a minimal element in $\mathscr{C}_X$.

A few warnings about minimal models:

- uniruled varieties do not have minimal models, since they carry free curves, on which the canonical class has negative degree;
- there exist smooth projective varieties which are not uniruled but which are not birational to any *smooth* projective variety with nef canonical bundle;[4]
- in dimension at least 3, minimal models may not be unique, although any two are isomorphic in codimension 1 ([D1, 7.18]).

Starting from X, Mori's idea is to try to simplify X by contracting K_X-negative extremal rays, hoping to end up with a variety X_0 which either has a fiber contraction (in which case X_0, hence also X, is covered by rational curves (see Section 4.5)) or has nef canonical divisor (hence no K_{X_0}-negative extremal rays). However, three main problems arise.

- The end-product of a contraction is usually singular. This means that to continue Mori's program, *we must allow singularities.* This is very bad from our point of view, since most of our methods do not work on singular varieties. Completely different methods are required.
- One must determine what kind of singularities must be allowed. Whichever choices we make, the singularities of the target of a small contraction are too severe and one needs to perform a flip. So we have the problem of *existence of flips.*
- One needs to know that the process terminates. The Picard number decreases for a fiber or divisorial contraction, but not for a flip! So we have the additional problem of *termination of flips:* do there exist infinite sequences of flips?

The first two problems have been overcome: the first one by the introduction of cohomological methods to prove the cone theorem on (mildly) singular varieties, the second one more recently in [BCHM] (see [Dr, cor. 2.5]). The third point is still open in full generality (see however [Dr, cor. 2.8]).

[4] This is the case for any desingularization of the quotient X of an abelian variety of dimension 3 by the involution $x \mapsto -x$ ([U, 16.17]); a minimal model here is X itself, but it is singular.

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