Lecture notes for the CIMPA school

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Contents

1 Introduction

To explain some of the main ideas of the Minimal Model Program and some of the tools used, we use some basic facts from graph theory. In particular, we describe a directed graph associated to the category of projective varieties. For this reason, we recall some of the basic definitions in graph theory.

Recall that a *directed graph* is a set of vertices connected by oriented edges, i.e. ordered pairs of vertices. Two edges are said to be *consecutive* if the ending vertex of one coincides with the starting vertex of the other. A *chain* in a directed graph is a sequence of distinct vertices connected by consecutive edges. A *cycle* in a directed graph is a sequence of consecutive ordered edges, starting and ending at the same vertex. A *tree* is a directed graph which does not contain any cycle. Note that the topological space underlying the tree is not necessarily simply connected, e.g. it could contain two distinct vertices and two edges connecting the two vertices with the same orientation.

Given two vertices X and Y of a directed graph, we say that Y is *below* X if we can find a chain starting from X and ending in Y. Clearly, If Y is below X, then we say that X is *above* Y. An *end-point* for a directed graph, is a vertex which does not admit any other vertex below it.

1.1 Projective surfaces

The easiest example in the study of directed graphs associated to the birational geometry of projective varieties is given by the category of smooth projective surfaces. To this end, we consider the *directed graph* whose vertices are smooth projective surfaces defined over an algebraically closed field k and whose edges are proper birational morphisms. The connected component containing a projective surface X corresponds to the birational class of X. We now look at some easy properties of this component. First, it is easy to check that this graph is a tree. Indeed, if X, Y are non-isomorphic projective surfaces connected by an edge, i.e. if there exists a non-trivial projective morphism $f\colon X \to Y$, then the second Betti number of X is greater than the one of Y. Thus, the claim follows easily.

Note that there are always infinitely many vertices above a vertex associated to a projective surface X, as it is always possible to blow-up an infinite sequence of points to obtain an infinite chain above X. On the other hand, using the inequality on the second Betti number described above, it is easy to check that starting from a vertex X, it is always possible to find an end-point below X. More specifically, there exists no infinite chain starting from X. Thus, we can think of the end-point Y to be a good representative of the connected class of X. We will see that also in higher dimension, one the main goals of the minimal model programme is to find the end-point of a connected component associated to a projective variety X.

We now show that projective surfaces can be divided into two large classes. The same dichotomy is expected to hold also in higher dimension.

First, we assume that X is a smooth projective surface such that $h^0(X, mK_X) > 0$ for some positive integer m. Then the subgraph obtained by considering the vertices below X and the corresponding edges is finite. In addition, there exists a unique vertex which is an end-point for the connected component containing X. Such a vertex Y is called the *minimal model* of X and, by Castelnuovo theorem, it is characterised by the fact that it does not admit any smooth rational curve E of self-intersection -1. Alternatively, Y is the only surface in the connected component of X such that K_Y is nef, i.e. $K_Y \cdot C \geq 0$ for any curve C in Y.

We now assume that X is a smooth projective surface such that $h^0(X, mK_X) = 0$ for all positive integer m. In this case, X is *uniruled*, i.e. it is covered by rational curves. It is possible to show that although the graph below X might be finite, there are always infinitely many end-points for the connected component of X. For example, if $X = \mathbb{P}^2$ is the two-dimensional projective space over the field k, then the connected component containing the vertex associated to X, corresponds to the set of all the smooth *rational surfaces*. Clearly, X is an end-point of such a graph, but also each Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$, with $n \in \mathbb{N}$, is such. Finally, note that not all the projective surfaces which admit a Mori fiber space is an end-point for the directed graph we have constructed (e.g. the blow-up of $\mathbb{P}^2$ at one point admits a Mori fiber space, but it corresponds to a vertex which is not an end-point).

1.2 Higher dimensional projective varieties

The goal of the minimal model program is to generalise the study of the directed graph we have seen in the previous section, to projective varieties of any dimension. Although we expect a similar behaviour, we will see that there are many problems arising as we go from dimension two to dimension three or higher.

In order to define the directed graph associated to projective varieties over an algebraically closed field k, we first try to understand what the edges are. A simple analysis in birational geometry shows that it is not suitable to consider only proper birational morphism within projective varieties, as otherwise we might get end-points of the graph which do not admit any special property and it would be hard to find a suitable characterisation of a good representative of the birational class of a projective variety. Thus, to solve this issue, we consider birational map with some extra properties. First, a birational map within normal projective varieties $\varphi\colon X \dashrightarrow Y$ is called a *contraction* if φ^{-1} does not contract any divisor. In other word, we do not want to consider maps like the blow-up of a proper subvariety, as this would not improve the understanding of our projective variety. Unfortunately, it is not enough to consider these maps as edges of our directed graph. Indeed, even in dimension three, it is possible to find pairs of non-isomorphic projective manifolds, which are isomorphic in codimension one, i.e. there exists X and Y which are isomorphic only after removing finitely many curves from both of them. In this case, X and Y would be part of a cycle, as the birational moprhism connecting X and Y and its inverse are both contraction. Therefore, we need to be more restrictive in the choice of those maps that define the edges of our directed graph.

Let $\varphi\colon X \dashrightarrow Y$ be a birational contraction. Then φ is said to be K-*negative* if there exist proper birational maps $p\colon W \to X$ and $q\colon W \to Y$ such that

$$p^*K_X = q^*K_Y + E$$

where E is an effective $\mathbb{Q}$-divisor and the support of E is equal to the union of all the exceptional divisors contracted by q[1]. It is then easy to show that if $\varphi\colon X \dashrightarrow Y$ is a K-negative birational contraction then its inverse is not K-negative. More in general, it is possible to check that if we define an edge of our graph to be a K-negative birational contraction, then the graph is a tree as it does not admit any cycle.

We now define the vertices of our new directed graph. Although, it would be tempting to consider only smooth projective varieties, as above it is possible to show that the end-points of our graph will not satisfy any useful property. On the other hand, we want to show that if the canonical divisor of a projective variety is nef then the vertex associated to X is an end-point of the graph. Indeed, we have:

Proposition 1. *Let X be a smooth projective variety such that K_X is nef. Then any K-negative birational contraction $\varphi\colon X \dashrightarrow Y$ is trivial, i.e. $\varphi = id_X$.*

Proof. Exercise. Hint: it follows from the Negativity Lemma below. □

Lemma 2 (Negativity of contraction)**.** *Let $p\colon W \to Z$ be a proper birational morphism of normal varieties. Let E be an effective p-exceptional $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on W. Then there is a component F of E which is covered by curves Σ such that $E \cdot \Sigma < 0$.*

Proof. Cutting by hyperplanes in W, we reduce to the case when W is a surface, in which case the claim reduces to the Hodge Index Theorem. E.g. see [2, Lemma 3.6.2] for more details. □

Thus, it is natural to ask under what condition, a projective variety corresponding to the end-point of the directed graph admits a nef canonical divisor. To this end, we need to consider projective varieties with some mild singularities. First, for simplicity, we assume that all the varieties we consider are $\mathbb{Q}$-*factorial*, i.e. we assume that X is normal and any Weil divisor S is such that mS is Cartier for some positive integer m. Moreover, we assume that X is *terminal*, which means that X admits a smooth variety Y above X. We now show that this is equivalent to the more classical definition:

Proposition 3. *X is terminal if and only if there exists a resolution $f\colon Y \to X$ such that*

$$K_Y = f^*K_X + E$$

for some $\mathbb{Q}$-divisor $E \geq 0$ whose support coincides with the exceptional locus of f.

It is easy to show that if X is terminal then the same property holds for any resolution of X.

[1]Note that this definition is slightly different than the one given in [2].

Proof. If $f\colon Y \to X$ is as in the proposition, then clearly X is terminal.

Let us assume now that there exists a smooth variety Z above X. Then there exist proper birational maps $p\colon W \to Z$ and $q\colon W \to X$ such that

$$p^*K_Z = q^*K_X + E$$

where E is an effective $\mathbb{Q}$-divisor and the support of E is equal to the union of all the exceptional divisors contracted by q. Let $h\colon \tilde{W} \to W$ be a resolution and let $p'\colon \tilde{W} \to Z$ and $q'\colon \tilde{W} \to X$ be the induced morphisms. Since Z is smooth, we may write

$$K_W = p'^*K_Z + G$$

for some $\mathbb{Q}$-divisor $G \geq 0$ whose support coincides with the exceptional locus of q'. Thus, we have

$$K_W = q'^*K_X + h^*E + G$$

and $h^*E + G \geq 0$. By assumption, the support of h^*E contains the strict transform of the exceptional locus of q in $\tilde{W}$ and the support of G contains the exceptional locus of h. Thus, the support of $h^*E + G$ coincides with the exceptional locus of q' and we are done. $\square$

Thus, terminal projective varieties appear naturally in the graph that we are considering. We can finally construct our directed graph: the vertices are terminal projective varieties defined over an algebraically closed field k and the edges are K-negative birational contractions.

Lemma 4. *Let X be a variety with terminal singularities. Then the singular locus of X has codimension at least* 3.

In particular, any terminal surface is smooth.

Proof. The case of surfaces is easy to check by considering a minimal resolution $h\colon \tilde{X} \to X$. In higher dimension, as in lemma 2, it is enough to cut by general hyperplanes. E.g. see [16, Corollary 5.18] for more details. $\square$

At this point, it is natural to ask if the same properties described in the case of surfaces, would hold for this directed graph. In particular, if X is a terminal projective variety, we can ask if there might exist an infinite chain starting from X. It is possible to show that this coincides with the following famous open problem:

Conjecture 5 (Termination of flips)**.** *Let X be a terminal projective variety. Then there exists no infinite sequence of flips*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \dots$$

starting from X

A *flip* is a special K-negative birational contraction which is an isomorphism in codimension one (see [16] for more details). Existence of flips for projective varieties defined over $\mathbb{C}$ was proven in [2, 10] and, later on, in [5, 7]. Termination of flips was proven by Shokurov in dimension three (cf. Theorem 12) and under some assumptions in higher dimension. In particular, one of the main achievements in this direction is obtained by combining the results in [1] and [12].

We now go back to the study of our directed graph and we can ask if the dichotomy within uniruled and non-uniruled surfaces extends to the case of higher dimensional projective varieties. More specifically, we expect that if $h^0(X, mK_X) > 0$ for some positive integer m, then there exists always an end-point below X, represented by a terminal projective variety Y with the property that K_Y is nef. As in the case of surfaces, Y will be called a *minimal model* of X.

Thus, we have the following:

Conjecture 6 (Existence of Minimal Models)**.** *Let X be a terminal projective variety with non-trivial canonical ring*

$$R(X, K_X) = \bigoplus_{m\geq 0} H^0(X, mK_X).$$

Then X admits a minimal model, i.e. there exists a K-negative birational map

$$\varphi\colon X \dashrightarrow Y$$

into a terminal projective variety Y, such that K_Y is nef.

If X is a complex variety of general type, i.e. if K_X is big, then the conjecture holds in any dimension [2] (cf. Theorem 20). Note that even in this case, Conjecture 5 is still open. Thus, the sub-graph of all the varieties below a given terminal projective variety of general type could contain an infinite chain, such that each vertex of this chain is connected to an end-point after finitely many consecutive edges.

If X is uniruled, then a very similar picture as in the case of surfaces holds. Although the graph given by the terminal projective varieties below X might be infinite, we do have a good description of the end-points of the connected component containing X. Indeed, these vertices correspond to varieties Y which admit a *Mori fibre space* [2] (cf. Theorem 22), i.e. a non-trivial map $\eta\colon Y \to Z$ onto a lower dimensional projective variety, such that the general fiber F is a (possibly singular) Fano variety, i.e. the anti-canonical divisor $-K_F$ is ample. Note that Y might admit more than one structure as a Mori fiber space (e.g. the simplest example is $\mathbb{P}^1 \times \mathbb{P}^1$).

Although the picture for uniruled varieties is quite well-understood, we still need to understand whether there is no other connected component of our directed graph, corresponding to projective varieties which do not belong to the two classes of varieties described above (i.e. non uniruled varieties with trivial canonical ring). Currently, this is one of the most important open problem in the Minimal Model Programme. For simplicity, we will only discuss it in the case of complex projective varieties:

Conjecture 7 (Weak Abundance Conjecture)**.** *Let X be a terminal complex projective variety with trivial canonical ring, i.e.*

$$R(X, K_X) = \mathbb{C}.$$

Then X is uniruled.

Note that in dimension three, all the conjectures described above (i.e. Conjecture 5, 6 and 7) hold in full generality for complex projective varieties (e.g. see [15, 16]).

2 Birational geometry of log pairs

It is immediate to see from the arguments in the previous section, that the canonical divisor plays a very important role in the study of the birational geometry of a projective variety over any algebraically closed field k. In addition, if $k = \mathbb{C}$, several generalisations of Kodaira's vanishing theorem, such as Kawamata-Viehweg vanishing, had a huge impact in birational geometry (e.g. in [5], it was shown that finite generation of the canonical ring follows almost directly from these results). This is another evidence of the fact that the canonical divisor is an essential tool in birational geometry. On the other hand, there are at least two main problems if we work in this generality. First, if S is a normal hypersurface in a projective variety X, then the adjunction formula [15] implies that

$$(K_X + S)|_S = K_S + \mathrm{Diff}_S$$

where Diff_S is an effective $\mathbb{Q}$-divisor on S, i.e. a linear combination of prime divisors of S with positive rational coefficients. Secondly, even in the simple case of a minimal elliptic surface $\pi\colon X \to C$ over a smooth curve C, we have that

$$K_X = \pi^*(K_S + D)$$

for some effective $\mathbb{Q}$-divisor D on C. Thus, it is natural to consider a larger category, which includes not only varieties but pairs (X, Δ) where X is a projective variety and Δ is an effective $\mathbb{Q}$-divisor. It is thus convenient to replace varieties by log pairs (X, Δ), and the canonical divisor by $K_X + \Delta$.

Therefore, our new goal is to define the right generalisation of the directed graph that we want to consider. To this end, we consider log pairs with Kawamata log terminal singularities: this is the most suitable condition in terms of the singularities of a log pair in the minimal model programme: a pair (X, Δ) is said to be *Kawamata log terminal* if for any proper birational morphism $\varphi\colon Y \to X$, we may write

$$K_Y + \Delta_Y = \varphi^*(K_X + \Delta)$$

where Δ_Y is a (non-necessarily effective) $\mathbb{Q}$-divisor whose coefficients are strictly less than 1. Note that this definition does not assume resolution of singularities.

We can now define the edges of our new directed graph. Let (X, Δ) and (Y, Δ) be two Kawamata log terminal pairs. An edge from (X, Δ) to (Y, Δ') is a birational contraction $f\colon X \dashrightarrow Y$ such that $f_*\Delta = \Delta'$ and f is $(K + \Delta)$-negative, i.e. there exist proper maps $p\colon W \to X$ and $q\colon W \to Y$ from a projective variety W which resolve the indeterminacy of f and such that

$$p^*(K_X + \Delta) = q^*(K_Y + \Delta') + E$$

where $E \geq 0$ is an effective $\mathbb{Q}$-divisor whose support coincides with the union of all the exceptional divisors of q. In particular, under these assumptions, if the log pair (Y, Δ') is below (X, Δ), then $H^0(X, m(K_X + \Delta)) \neq 0$ for some positive integer m if and only if the same property holds for (Y, Δ').

Thus, as in the absolute case (i.e. when $\Delta = 0$), given a Kawamata log terminal pair (X, Δ), we can investigate the graph given by pairs below (X, Δ). In particular, it is expected that there are no infinite chains below (X, Δ). As in Conjecture 5, this corresponds to *termination of log-flips*. In other words, it is expected that there exists always an end-point below (X, Δ). If $H^0(X, m(K_X + \Delta)) \neq 0$ for some positive integer m, then an end-point (Y, Γ) is characterised by the property that $K_Y + \Gamma$ is nef. The pair (Y, Γ) is called a *minimal model* of (X, Δ). In [2], it was proven that if Δ is big, then (X, Δ) always admits a minimal model (cf. Theorem 23).

We now consider the case of Kawamata log terminal pairs (X, Δ) such that $H^0(X, m(K_X + \Delta)) = 0$ for all positive integers m. Similarly as in the absolute case, it is expected that an end-point (Y, Δ') below (X, Δ) admits a *Mori fiber space*, which is a non-trivial morphism $\eta\colon Y \to Z$ such that $-(K_X + \Delta)|_F$ is ample for the general fiber F of η. In [2], it was proven that if $(K_X + \Delta)$ is not pseudo-effective (i.e. if there exists an ample $\mathbb{Q}$-divisor such that $K_X + \Delta + A$ is not big) then there exists an end-point below (X, Δ), which admits a Mori fiber space.

The main tools used in the study of log pairs are just generalisations of the results in the absolute case. E.g. Kodaira vanishing generalises to:

Theorem 8 (Kawamata-Viehweg Vanishing Theorem)**.** *Let (X, Δ) be a complex $\mathbb{Q}$-factorial* *projective Kawamata log terminal pair. Let N be a Weil divisor such that*

$$N \equiv \Delta + A$$

where A is a big and nef $\mathbb{Q}$-divisor.

Then $H^i(X, \mathcal{O}_X(K_X + N)) = 0$ for all $i > 0$.

Proof. E.g. see [16, Theorem 2.70]. $\square$

Furthermore, we have:

Theorem 9 (Base point free theorem)**.** *Let (X, Δ) be a complex $\mathbb{Q}$-factorial projective Kawamata log terminal pair. Let D be a nef $\mathbb{Q}$-divisor such that*

$$D = K_X + \Delta + A$$

where A is a big and nef $\mathbb{Q}$-divisor.

Then D is semi-ample, i.e. there exists a positive integer m such that mD is Cartier and $|mD|$ is base point free.

Proof. E.g. see [16, Theorem 3.3]. $\square$

Theorem 10 (Cone Theorem)**.** *Let (X, Δ) be a complex projective Kawamata log terminal pair. Then there are countably many rational curves $C_1, C_2, \ldots$ such that*

$$0 < -(K_X + \Delta) \cdot C_i \leq 2 \dim X$$

and

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\Delta)\geq 0} + \sum \mathbb{R}_{\geq 0}[C_j].$$

Proof. E.g. see [16, Theorem 3.7]. $\square$

Example 11. Assume that (X, Δ) is a complex log Fano pair (i.e. (X, Δ) is Kawamata log terminal and $-(K_X + \Delta)$ is big and nef). Let D be a nef divisor. Then, we may write

$$D = K_X + \Delta + (D - (K_X + \Delta))$$

and since $D - (K_X + \Delta)$ is big and nef, Theorem 9 implies that D is semi-ample.

Now assume that S is the blow-up of $\mathbb{P}^2$ at nine very general points and let E be the unique elliptic curve in $\mathbb{P}^2$ passing through these points. Then $-K_S = C$ where C is the strict transform of E in S and in particular, $-K_S$ is not semi-ample, as C is the contained in the base locus of $|m(-K_S)|$ for any positive integer m. (the problem is that $-K_S$ is not big and therefore S is not log Fano).

Finally, let H be an ample Cartier divisor and let $Z = \mathbb{P}(\mathcal{O}_S \oplus \mathcal{O}_S(-H))$ with projection map $p\colon Z \to S$. Let ξ the tautological class on Z and let $\Delta = \xi$.

Then, it is easy to check that

$$-(K_Z + \Delta) = -p^*K_S + \xi + p^*H$$

is big and nef. On the other hand, $D = -(K_Z + \Delta)$ is not semi-ample. Indeed

$$D|_S = -K_S$$

and we showed that $-K_S$ is semi-ample (see [9, Example 5.2] for more details). Note that (X, Δ) is log smooth but it is not Kawamata log terminal (the coefficient of Δ is equal to one).

We now consider a special case, which illustrates the fact that the point of view of directed graphs is useful to understand the birational geometry of a Kawamata log terminal pair (X, Δ). Assume that Δ is big and that $K_X + \Delta$ is not pseudo-effective. Since Δ is big, the non-vanishing theorem proven in [2] implies that the assumption that $K_X + \Delta$ is not pseudo-effective coincides with the a-priori stronger assumption that $H^0(X, m(K_X + \Delta)) = 0$ for any positive integer m. Thus, it is natural to ask if (X, Δ) admits infinitely many end-points below (X, Δ). Note that the picture in the absolute case (i.e. when $\Delta = 0$) is very different, as we have already showed that there might be infinitely many edges starting from the vertex associated to $(X, 0)$. On the other hand, assuming that Δ is big, it is possible to show, by using boundedness of the length of extremal rays [13], that there are only finitely many edges starting from (X, Δ). Thus, König's Lemma implies that the sub-graph given by all the vertices below (X, Δ) is finite if and only if there are no infinite chains starting from (X, Δ) (see [17, Lemma 6.7] for more details). Clearly, if the sub-graph below (X, Δ) is finite, there are only finitely many end-point below (X, Δ).

3 Termination of threefold flips

We now want to show how the study of singularities imply termination of threefold flips. In particular, we want to show:

Theorem 12 (Shokurov)**.** *There exists no infinite sequence of three dimensional K-negative birational contractions.*

First, we need to define the log discrepancy of a variety with respect to a divisorial valuation. To this end, we consider the more general case of a log pair (X, Δ).

Definition 13. Let (X, Δ) be a log pair and let $p\colon Y \to X$ be a proper birational morphism. Then we may write

$$K_Y + \Delta_Y + \sum (1 - a_i) E_i = p^*(K_X + \Delta)$$

where Δ_Y is the strict transform of Δ on Y (i.e. $\Delta_Y = p_*^{-1}\Delta$) and the sum is taken over all the p-exceptional divisor E_i of p. The rational number $a_i = a(E_i, X, \Delta)$ denotes the *log discrepancy* of (X, Δ) with respect to E_i.

If $\Delta = 0$, then we denote $a(E, X, \Delta)$ simply by $a(E, X)$.

Remark 14. Note that, in many other references, such as in [16], the same symbol $a(E, X, \Delta)$ denotes the discrepancy of (X, Δ) with respect to E, which is, according to our notation, nothing but $a(E, X, \Delta) - 1$.

It is important to understand that the log discrepancy does not depend on the morphism p that we consider. In other words, using the same notation as above, if E is a p-exceptional divisor and $q\colon Z \to X$ is another proper birational morphism such that the induced birational map $\varphi\colon Y \dashrightarrow Z$ is an isomorphism at the general point of E then $a(E, X, \Delta)$ coincides with $a(\varphi_* E, X, \Delta)$ and therefore it can be computed on Z. It is a good exercise to check that the equality above holds. Thus, given a log pair (X, Δ), we can define $a(E, X, \Delta)$ with respect to any divisorial valuation E over X.

The idea is that the smaller is the minimum value of $a(E, X, \Delta)$ with respect to all the divisorial valuations E over X, and the more singular the log pair (X, Δ) is. In particular, we have:

Lemma 15. *A log pair (X, Δ) is Kawamata log terminal if and only if for any proper birational morphism $p\colon Y \to X$ and for any p-exceptional divisor E, we have $a(E, X, \Delta) > 0$.*

A variety X is terminal if and only if for any proper birational morphism $p\colon Y \to X$ and for any p-exceptional divisor E, we have $a(E, X) > 1$.

Proof. Exercise. □

Example 16. Let X be a smooth threefold and let $p\colon Y \to X$ be the blow-up of X along a smooth curve C, with exceptional divisor E. Then $a(E, X) = 2$.

Lemma 17. *Let X be a terminal variety. Then there are at most finitely many divisorial valuations E such that $a(E, X) < 2$.*

Proof. Let $f\colon Y \to X$ be a resolution. Then, we may write

$$K_Y = f^* K_X + \Gamma \tag{1}$$

where $\Gamma \geq 0$ is a f-exceptional divisor. Let E be a divisorial valuation over X such that $a(E, X) < 2$ and let us assume that E is not a divisor over Y. Then, (1) implies that

$$a(E, Y) \leq a(E, X) < 2.$$

Since Y is smooth, we easily obtain a contradiction. Thus, E is a divisor on Y. Since there are only finitely many divisors on Y which are contracted by f, the claim follows immediately.

For a more general result, see [16, Proposition 2.36]. □

We define the *difficulty* of a terminal three-fold as

$$d(X) = \#\{E \mid a(E, X) < 2\}.$$

By Lemma 17, $d(X)$ is a non-negative integer.

Example 18. It is easy to check that if X is a smooth or Gorenstein (i.e. K_X is Cartier) terminal variety, then $d(X) = 0$. The opposite is also true, at least over $\mathbb{C}$, i.e. if $d(X) = 0$ then X is Gorenstein. But its proof relies on the classification of terminal singularities.

Lemma 19. *Let $\varphi\colon X \dashrightarrow Y$ be a K-negative birational contraction between terminal three-folds which is an isomorphism in codimension one and let E be a divisorial valuation over X such that $a(E, Y) < 2$.*

Then $a(E, X) < 2$ and, in particular,

$$d(X) \geq d(Y).$$

More in general, if $X \dashrightarrow Y$ is a K-negative birational contraction between terminal three-folds, then $d(X) \geq d(Y) - \rho(X/Y)$.

Proof. By assumption, there exist proper birational maps $p\colon W \to X$ and $q\colon W \to Y$ such that

$$p^*K_X = q^*K_Y + F, \tag{2}$$

where F is an effective q-exceptional $\mathbb{Q}$-divisor. Let E be a divisorial valuation over Y such that $a(E, Y) < 2$. After possibly replacing W by a variety W' which admits a proper birational morphism $W' \to W$, we may assume that W is smooth and that E is a divisor on W (note that we cannot assume anymore that the support of F is equal to the union of all the exceptional divisors contracted by q as in our definition of K-negative birational contraction, but we do not need this fact here). There are two cases: either E is a divisor on X which is contracted by φ or it is contracted by p. If φ is an isomorphism in codimension one, then only the second case can occur and (2) easily implies that $a(E, X) < 2$. Thus, the claim follows. □

We can finally prove Theorem 12.

Proof. Assume that

$$X = X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \dots$$

is an infinite sequence of K-negative birational contraction in dimension three. Then, for each i, we have

$$\rho(X_i) \geq \rho(X_{i+1})$$

where the equality holds if and only if $X_i \dashrightarrow X_{i+1}$ is an isomorphism in codimension one. Since $\rho(X_i)$ is a positive integer, after possibly taking a subsequence, we may assume that each $X_i \dashrightarrow X_{i+1}$ is an isomorphism in codimension one. It is enough to show that $d(X_i) > d(X_{i+1})$.

Lemma 19 implies that if E is a divisorial valuation over X_{i+1} such that $a(E, X_{i+1}) < 2$ then $a(E, X_i) < 2$. Thus, it is enough to construct a divisorial valuation E over X_i such that $a(E, X_i) < 2$ and $a(E, X_{i+1}) \geq 2$.

By assumption, there exist proper birational maps $p\colon W \to X_i$ and $q\colon W \to X_{i+1}$ such that

$$p^* K_{X_i} = q^* K_{X_{i+1}} + G, \tag{3}$$

where G is an effective $\mathbb{Q}$-divisor whose support is equal to the union of all the exceptional divisors contracted by q.

Lemma 4 implies that the singularities of X_{i+1} are isolated. On the other hand, one can check that there exists a divisor F on W such that $\xi = q(F)$ is a curve in X_{i+1}. Let $X'_{i+1} \to X_{i+1}$ be the blow-up of X_{i+1} along ξ and let E be the exceptional divisor. Then E is a divisorial valuation on X_{i+1} such that $a(E, X_{i+1}) = 2$ (see Example 16). On the other hand, (3) implies that $a(E, X_i) < 2$ and the claim follows.

Thus, $d(X_i) > d_(X_{i+1})$ and the sequence must terminate. □

Note that the same proof, combined with Example 18, implies that if X is Gorenstein, then there are no non-trivial K-negative birational contraction $X \dashrightarrow Y$ in dimension three. In particular, if X is Gorenstein of dimension three, then there are no flips $X \dashrightarrow Y$.

4 Some results

Some of the main results in [2] are:

Theorem 20. *Let X be a smooth complex projective variety of general type. Then X admits a minimal model $X \dashrightarrow Y$.*

Theorem 21. *Let X be a smooth complex projective variety. Then the canonical ring*

$$R(X, K_X) = \bigoplus_{m \geq 0} (X, mK_X)$$

is finitely generated.

Theorem 22. *Let X be a uniruled smooth complex projective variety. Then X admits a Mori fibre space $X \dashrightarrow Y$.*

We first want to show that the three results above follow from the following

Theorem 23. *Let (X, Δ) be a Kawamata log terminal pair over $\mathbb{C}$ such that Δ is a big $\mathbb{Q}$-divisor and $K_X + \Delta$ is pseudo-effective. Then (X, Δ) admits a minimal model $X \dashrightarrow Y$.*

Recall, that a $\mathbb{Q}$-divisor D is pseudo-effective if, given any ample $\mathbb{Q}$-divisor A, we have that $D + A$ is big. A minimal model for a pair (X, Δ) is a $(K_X + \Delta)$-negative birational contraction $\varphi\colon X \dashrightarrow Y$ such that $K_Y + \varphi_* \Delta$ is nef.

We first show how Theorem 23 implies the three results above:

Proof of Theorem 20. By assumption K_X is big. In particular, there exists a positive integer m such that $|mK_X| \neq \emptyset$. Let $D \in |mK_X|$. Since X is smooth, it is easy to check that if ε is a sufficiently small rational number and $\Delta = \varepsilon D$, then (X, Δ) is Kawamata log terminal. Note that, since

$$K_X + \Delta = (1 + \varepsilon m) K_X \tag{4}$$

it follows that $K_X + \Delta$ is big and, in particular, it is pseudo-effective. Thus, Theorem 23 implies that $(K_X + \Delta)$ admits a minimal model $\varphi\colon X \dashrightarrow Y$. It is then easy to check that (4) implies that φ is also a minimal model for X. □

Proof of Theorem 21. We first consider a slightly different set-up. We assume that (X, Δ) is a Kawamata log terminal pair such that Δ is big. We want to show that the ring

$$R(X, K_X + \Delta) = \bigoplus_{m \text{ suff. divisible}} H^0(X, m(K_X + \Delta))$$

is finitely generated. Clearly if $H^0(X, m(K_X + \Delta)) = 0$ for any m then $R(X, K_X + \Delta)$ is finitely generated. Thus, we may assume that $H^0(X, m(K_X + \Delta)) = 0$ for some m and in particular $K_X + \Delta$ is pseudo-effective. Theorem 23 implies that (X, Δ) admits a minimal model $\varphi\colon X \dashrightarrow Y$ and, since φ is $(K_X + \Delta)$-negative, it follows that $R(X, K_X + \Delta) \simeq R(Y, K_Y + \varphi_* \Delta)$. Thus, after replacing X by Y and Δ by $\varphi_* \Delta$, we may assume that $K_X + \Delta$ is nef. Since Δ is big, we can find an ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor E such that

$$\Delta \simeq A + E.$$

Let $\varepsilon > 0$. We may write

$$\Delta = (1-\varepsilon)\Delta + \epsilon E + \epsilon A$$

and since (X, Δ) is Kawamata log terminal, it follows that if ε is sufficiently small, then

$$(X, (1-\varepsilon)\Delta + \epsilon E)$$

is also Kawamata log terminal. We have,

$$K_X + \Delta = K_X + (1-\varepsilon)\Delta + \epsilon E + \epsilon A$$

and since ϵA is ample, the base point free theorem (cf. Theorem 9) implies that $K_X + \Delta$ is semi-ample. Thus, there exists a morphism $\eta\colon X \to Z$ such that $K_X + \Delta = \eta^* H$ for some ample $\mathbb{Q}$-divisor H. It is easy to check, that up to truncation the ring $R(X, K_X + \Delta)$ is isomorphic to the ring

$$R(Z, H) = \bigoplus_{m \text{ suff. divisible}} H^0(X, mH)$$

which is finitely generated since it is the quotient of a polynomial ring.

We now go back to our set-up and we assume that X is a smooth projective variety. As above, we may assume that the Kodaira dimension of X is non-negative (otherwise there is nothing to prove). Let $\psi\colon X \dashrightarrow Y$ be the Iitaka fibration associated to K_X. Then, by a result of Fujino and Mori [8], it follows that, up to truncation, the canonical ring $R(X, K_X)$ of X is isomorphic to the ring associated to a Kawamata log terminal pair (Y, Γ) for some big $\mathbb{Q}$-divisor Γ on Y. Thus, the result follows from the claim above. $\square$

Sketch of the proof of Theorem 22. Since X is uniruled, it follows that K_X is not pseudo-effective. In particular, if A is an ample $\mathbb{Q}$-divisor on X then there exists $t > 0$ such that $K_X + tA$ is not big. Let

$$\lambda = \sup\{t > 0 \mid K_X + tA \text{ is not big}\}.$$

Then $K_X + \lambda A$ is pseudo-effective but not big. It is easy to show that there exists $\Delta \sim_{\mathbb{Q}} \lambda A$ such that (X, Δ) is Kawamata log terminal. Thus, by Theorem 23, we may find a proper birational contraction $\varphi\colon X \dashrightarrow Y$ such that $K_Y + \varphi_*\Delta$ is nef. Since $K_X + \lambda A$ is not big, it follows that $K_Y + \varphi_*\Delta$ is also not big. Since $\varphi_*\Delta$ is big, as in the proof of Theorem 21, it follows from the base point free theorem[2] that $K_Y + \varphi_*\Delta$ is semi-ample and in particular, there exists $\eta\colon Y \to Z$ such that

$$K_Y + \varphi_*\Delta = \eta^* H \tag{5}$$

[2]Note that a priori λ might be an irrational number. On the other hand, Theorem 9 holds in a more general context, assuming that Δ is a $\mathbb{R}$-divisor rather than a $\mathbb{Q}$-divisor

for some ample $\mathbb{Q}$-divisor H on Z. In particular $\dim Z < \dim Y = \dim X$.

We now assume that the restriction of $\varphi_*\Delta$ to the general fibre of η is ample (e.g. this is true if $\rho(Y/Z) = 1$). Then, since the restriction of η^*H to the general fibre of η is numerically trivial, it follows from (5), that the general fibre of Y is Fano. Thus Y is a Mori fibre space. The general case (without assuming that the restriction of $\varphi_*\Delta$ to the general fibre of η is ample) is slightly harder. $\square$

5 Minimal Model Program with Scaling

We now illustrate some of the steps to prove Theorem 23. We begin by considering the classical minimal model program, due to Mori.

We start with a complex projective manifold X. We want to find a K-negative birational contraction $\varphi\colon X \dashrightarrow Y$ such that, either Y is minimal or it admits a Mori fibre space.

If K_X is nef, then we are done, as X is minimal and we can stop here. Thus, we may assume that K_X is not nef. By the cone theorem (cf. Theorem 10), it is easy to show that there exists an ample $\mathbb{Q}$-divisor such that $K_X + A$ is nef but not ample, and there exists a rational curve ξ such that, $K_X \cdot \xi < 0$ and for any curve C in X, we have that

$$(K_X + A) \cdot C = 0 \qquad \text{if and only if } [C] \in \mathbb{R}_+[\xi].$$

Since A is ample and $K_X + A$ is nef, the base point free theorem (cf. Theorem 9) implies that there exists $f\colon X \to Z$ with connnected fibers and such that $K_X \cdot C < 0$ for any curve C s.t. $f(C)$ is a point. More precisely, C is contracted if and only if $[C] \in \mathbb{R}_+[\xi]$.

We now distinguish three cases:

1) if $\dim Z < \dim X$ then, X is automatically a Mori fibre space, and we can stop here.

2) if f is birational and the exceptional locus of f is a divisor then we just replace X by Z and we start again. Note that f is automatically a K-negative birational contraction. On the other hand, X is not anymore necessarily smooth, but it is always terminal and all the steps above would work without any change.

3) Otherwise, the morphism f is a flipping contraction. In this case, Z is too singular and we cannot replace X by Z. Indeed, it is possible to show that K_Z is never $\mathbb{Q}$-Cartier and pretty much everything above would not hold true anymore. On the other hand, [10, 2] imply that there exists a flip $\varphi\colon X \dashrightarrow X^+$ of ξ. It follows again that X^+ is terminal and φ is a K-negative birational contraction. Thus, w ecan replace X by X_+ and start all over again.

At this point, it only remains to show that this process would terminate after finitely many steps, so that we obtain either a minimal model or a Mori fibre space which is birational to X. Note that 2) would only occur finitely many times, because, as we mentioned before, after each such step, the Picard number of X would decrease by one (and any time that case 3) occurs, the Picard number would not change). Thus, the only (big) problem is to show that 3) does not occur infinitely many times. Theorem 12 solves this problem in dimension 3, but in higher dimension, this is still an open question.

The idea of the minimal model program with scaling is that, instead of choosing a random sequence of flips, we show that there exist a special sequence which terminates. To this end, it turns out that it is more natural to work in the more general context of log pairs (X, Δ). We begin by assuming that (X, Δ) is log canonical and we pick a $\mathbb{Q}$-divisor C such that $K_X + \Delta + C$ is nef and the pair $(X, \Delta + C)$ is log canonical (e.g. we may choose A to be a sufficiently ample divisor, and for any sufficiently large, we define $C = \frac{1}{m} B$ where $B \in |mA|$ is a general element). We define

$$\lambda = \min\{t \in \mathbb{R}_{\geq 0} |\ K_X + \Delta + t\ C \text{ is nef}\}.$$

Note that if $\lambda = 0$ then $K_X + \Delta$ is nef and we can stop here. Thus, we may assume that $\lambda > 0$. Then by the cone theorem (cf. Theorem 10) there exists a rational curve ξ, such that $\mathbb{R}_+[\xi]$ is an extremal ray for $\overline{N}E(X)$ and such that

$$\begin{aligned} &(K_X + \Delta) \cdot \xi < 0 \qquad \text{and} \\ &(K_X + \Delta + \lambda C) \cdot \xi = 0. \end{aligned}$$

As before, we might find a morphism $f \colon X \to Y$ which contracts any curve which is numerically equivalent to a multiple of ξ. If f defines a Mori fibre space, then we are done. Thus, we may assume that f is birational. If f is a divisorial contraction then we denote Y by X', otherwise f is a flipping contraction and we consider the flip $X \dashrightarrow X'$ associated to X. In both cases, if Δ' and C' denote the strict trasnsform on X' of Δ and C respectively, then it is easy to check that (X', Δ') is log canonical and $K_{X'} + \Delta' + \lambda C'$ is nef. We now consider

$$\lambda' = \min\{t \in \mathbb{R}_{\geq 0} |\ K_{X'} + \Delta' + t\ C' \text{ is nef}\} \leq \lambda$$

and we proceed as above. Thus, we obtain a sequence of $(K + \Delta)$-negative birational contractions

$$X = X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \dots$$

but also a decreasing sequence of rational numbers

$$\lambda = \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \dots ...$$

Note that the choice of contractions depends on the initial choice of the $\mathbb{Q}$-divisor C. For this reason, this process, is called a $(K_X + \Delta)$-*Minimal Model Program with scaling of* C.

We now present some of the main ingredients in the proof that, if (X, Δ) is Kawamata log terminal and Δ is big, then the minimal model program with scaling terminates. Some of the details are rather technical and we refer to [2].

We first need to enlarge the category of Kawamata log terminal pairs, to a slightly larger (and more suitable) one. A log pair (X, Δ) is divisorially log terminal if there exists a log resolution ϕ such that $a(E, X, \Delta) > 0$ for any ϕ-exceptional divisor E.

It is a good exercise to show that if (X, Δ) is dlt and A is ample, there exists $\Delta' \sim_{\mathbb{Q}} \Delta + A$ such that (X, Δ') is klt.

Before we proceed, we need the following result, due to Shokurov

Theorem 24 (Special termination)**.** *Assume that the main results of the Minimal Model Program hold in dimension* $\leq n-1$. *Assume that* (X, Δ) *is a divisorially log terminal pair of dimension* n. *Let*

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \dots$$

be a sequence of $(K + \Delta)$*-flips. Let* $S = \llcorner \Delta \lrcorner$ *and let* S_i *be the strict transform of* S *on* X_i.

Then, if $i \gg 0$, *the flipping locus of* $X_i \dashrightarrow X_{i+1}$ *is disjoint from* S_i.

Proof. We omit the proof. It uses similar ideas as Theorem 12. See [, Theorem 4.2.1] for more details. $\square$

We begin by considering a very special case:

Lemma 25 (Key Lemma)**.** *Let* (X, Δ) *a log pair and let* $C, D \geq 0$ *be effective* $\mathbb{Q}$*-divisors such that*

(1) $\Delta = S + A + B$, *where* $\llcorner \Delta \lrcorner = S$, $A \geq 0$ *is an ample* $\mathbb{Q}$*-divisor and* $B \geq 0$,

(2) $(X, \Delta + C)$ *is divisorially log terminal (in particular* S *and* $\operatorname{Supp} C$ *do not have any common component) and* $K_X + \Delta + C$ *is nef, and*

(3) the support of D *is contained in* S *and there exists* $\alpha \geq 0$ *such that*

$$K_X + \Delta \sim D + \alpha C.$$

Then the $(K_X + \Delta)$*-Minimal model program with scaling of* C *terminates.*

Sketch of the proof. We define λ as above, and we consider a rational curve ξ such that

$$(K_X + \Delta) \cdot \xi < 0 \qquad \text{and}$$
$$(K_X + \Delta + \lambda C) \cdot \xi = 0.$$

In particular, it follows immediately that $C \cdot \xi > 0$. Thus, (3) implies that $D \cdot \xi < 0$ and therefore ξ is contained in the support of D. Applying (3) again, it follows that ξ is contained in S. Thus, special termination (cf. Theorem 24) implies termination for the MMP with scaling and we are done. □

Note that special termination holds under the assumption that all the results of the minimal model program hold true in dimension lower than the dimension of X. On the other hand, it is possible to use a "special" version of special termination which holds unconditionally.

We now present the main ideas used in the proof of Theorem 23. Let (X, Δ) be a dlt pair such that Δ is big and $K_X + \Delta$ is dlt. It is possible (but not easy) to show that there exists a $\mathbb{Q}$-divisor D such that $K_X + \Delta \sim_{\mathbb{Q}} D \geq 0$. We further assume that the pair $(X, \Delta + D)$ is log smooth.

We proceed by induction on the following positive integer

$$k := \# \text{ of irreducible components of } D \text{ which are not contained in } \llcorner \Delta \lrcorner$$

We first consider the case $k = 0$. This means that all the support of D is contained in $\llcorner \Delta \lrcorner$. In this case, we take an ample divisor H such that $K_X + \Delta + H$ is ample. Thus, Lemma 25 immediately implies that the pair (X, Δ) admits a minimal model and we are done.

We now assume that $k > 0$. Then we may write

$$D = D_1 + D_2 \qquad \text{with } D_1, D_2 \geq 0$$

where the components of D_2 are exactly all the components of D which are not contained in $\llcorner \Delta \lrcorner$. We define:

$$\lambda := \sup\{\, t \in [0,1] \,|\, (X, \Delta + tD_2) \text{ is log canonical} \,\}.$$

By assumption, we have $\lambda > 0$ and it follows that the number of irreducible components of $D + \lambda D_2$ which are not contained in $\llcorner \Delta + \lambda D_2 \lrcorner$ is less than k. Thus, by induction, it follows that the pair $(X, \Delta + \lambda D_2)$ admits a minimal model $X \dashrightarrow X'$. After replacing X by X', and with a bit of work [3], it is possible to assume that if

$$\Theta = \Delta + \lambda D_2$$

[3]the main idea, on why we can do this, relies on the fact that any divisor contracted by the birational contraction $X \dashrightarrow X'$ in contained in the stable base locus of $K_X + \Delta$

then $K_X + \Theta$ is nef. Let $C = \lambda D_2$. We have

$$K_X + \Delta \sim_{\mathbb{Q}} D_1 + \frac{1}{\lambda} C$$

and the support of D_1 is contained in $\lfloor \Delta \rfloor$. Thus, we may apply Lemma 25 again and we obtain that (X, Δ) admits a minimal model.

As a final remark, note that Theorem 23 can be proven as a consequence of finite generation [5, 7]. We refer to [4] for a short survey.

6 Flops and Sarkisov Program

We have seen that in general projective varieties X (and Kawamata log terminal pairs (X, Δ)) are divided into two large families, depending on the existence of a global section of mK_X (respectively $m(K_X + \Delta)$) for some positive integer m. If there is no such a section, than the picture is more complicated, as the vertices might have infinitely many edges starting from it.

Now it is natural to investigate the relations within the end-points of the directed graphs we constructed. First, assume that X is an end-point in the absolute case (i.e. with $\Delta = 0$), represented by a terminal projective variety such that K_X is nef. Then by a result of Kawamata [14], any other end-point Y in the connected component of X is connected to X by a composition of flops (see [16] for a definition of flop). Note that a flop $\varphi \colon X \dashrightarrow Y$ is a special K-trivial isomorphism in codimension one, i.e. if $p \colon W \to X$ and $q \colon W \to Y$ are proper morphisms which resolve the indeterminacy locus of φ, then

$$p^* K_X = q^* K_Y.$$

A similar picture holds for log pairs (see [14] for more details).

It is conjectured that any two terminal projective varieties which are isomorphic in codimension one and K-equivalent (i.e. they admit a K-trivial birational map which is an isomorphism in codimension one) are connected by a finite sequence of flops (e.g. see [18]). At the moment, the conjecture is open even in the case of smooth projective varieties of dimension three.

Finally, if X is a terminal projective variety of general type, then the number of end-points in the connected component of the graph containing X is always finite [2].

We now consider the case of uniruled projective varieties. We have seen that even in dimension two, the number of end-points in each connected component of the graph which contains a uniruled projective variety is infinite. It is therefore natural to ask about the relation within birational pairs of projective varieties which admit a Mori fiber space. More specifically let $\eta \colon X \to Z$ and $\eta' \colon X \to Z'$ be two Mori fiber spaces, with X and X' terminal projective varieties which admit a birational map $\psi\colon X \dashrightarrow Y$. Then the goal of the Sarkisov programme is to show that φ can be decomposed into a sequence of *Sarkisov links*, which are elementary transformations obtained as compositions of flops and divisorial contractions (e.g. see [11] for more details). The programme was successfully carried out in [6, 3] in dimension three and in [11] in full generality.

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